[DAC61333] KALKULUS LANJUT
"Turunan Fungsi Dua Variabel atau Lebih"
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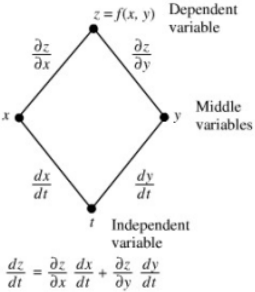
6. Aturan Rantai

6.1 Aturan Rantai Pertama

Theorem

Misalkan $x = x(t)$ *dan* $y = y(t)$ *terturunkan di* t *dan misalkan* $z = f(x, y)$ *terturunkan di* $(x(t), y(t))$, *maka* $z = f(x(t), y(t))$ *dapat diturunkan di* t *dan*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$



6.1 Aturan Rantai Pertama

Example

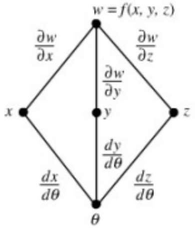
Andaikan $z = x^2 y^3$ dengan $x = t^2 + 1$ dan $y = t^2 - 1$, hitunglah dz/dt.

Solution

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \\
&= \left(2xy^3\right)(2t) + \left(3x^2y^2\right)(2t) \\
&= 4t\left(t^2+1\right)\left(t^2-1\right)^3 + 6t\left(t^2+1\right)^2\left(t^2-1\right)^2 \\
&= 4t\left(t^4-1\right)\left(t^2-1\right)^2 + 6t\left(t^4-1\right)^2 \\
&= 2t\left(t^4-1\right)\left[2\left(t^2-1\right)^2 + 3\left(t^4-1\right)\right] \\
&= 2t\left(t^4-1\right)\left(5t^4-4t^2-1\right)
\end{aligned}$$

6.1 Aturan Rantai Pertama

Aturan rantai untuk kasus 3 variabel



$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta}$$

Example

Andaikan $w = x^2y + y + xz$ dengan $x = \cos\theta$, $y = \sin\theta$ dan $z = \theta^2$, carilah $dw/d\theta$ dan hitung nilainya di $\theta = \pi/3$.

6.1 Aturan Rantai Pertama

Solution

$$\begin{aligned}
\frac{dw}{d\theta} &= \frac{\partial w}{\partial x}\cdot\frac{dx}{d\theta}+\frac{\partial w}{\partial y}\cdot\frac{dy}{d\theta}+\frac{\partial w}{\partial z}\cdot\frac{dz}{d\theta} \\
&= (2xy+z)(-\sin\theta)+\left(x^2+1\right)(\cos\theta)+(x)(2\theta) \\
&= -(2xy+z)(\sin\theta)+\left(x^2+1\right)(\cos\theta)+2x\theta \\
&= -2\cos\theta\sin^2\theta-\theta^2\sin\theta+\cos^3\theta+\cos\theta+2\theta\cos\theta
\end{aligned}$$

nilainya di $\theta = \pi/3$

$$\begin{aligned}
\frac{dw}{d\theta} &= 2\cos\theta\sin^2\theta-\theta^2\sin\theta+\cos^3\theta+\cos\theta+2\theta\cos\theta \\
&= 2\cos\frac{\pi}{3}\sin^2\frac{\pi}{3}-\left(\frac{\pi}{3}\right)^2\sin\frac{\pi}{3}+\cos^3\frac{\pi}{3}+\cos\frac{\pi}{3}+2\theta\cos\theta \\
&= -\frac{1}{8}-\frac{\sqrt{3}\pi^2}{18}+\frac{\pi}{3}
\end{aligned}$$

6.2 Aturan Rantai Kedua

Theorem

Misalkan $x = x(s,t)$ *dan* $y = y(s,t)$ *mempunyai turunan-turunan parsial pertama di* (s,t) *dan misalkan* $z = f(x,y)$ *terturunkan di* $(x(s,t), y(s,t))$, *maka* $z = f(x(s,t), y(s,t))$ *mempunyai turunan-turunan parsial pertama yang diberikan oleh*

$$(1) \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \qquad (2) \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Example

Jika $z = 3x^2 - y^2$ dengan $x = 2s + 7t$ dan $y = 5st$, carilah $\partial z/dt$ dan nyatakan dalam bentuk s dan t.

6.2 Aturan Rantai Kedua

Solution

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\\ &= (6x)(7)+(-2y)(5s)\\ &= 42x-10sy\\ &= 42(2s+7t)-10s(5st)\\ &= 84s+294t-50s^2t\end{aligned}$$

Example

Jika $w = x^2 + y^2 + z^2 + xy$ dengan $x = st, y = s - t$ dan $z = s + 2t$, carilah

$$\frac{\partial w}{\partial t}\Big|_{s=1,t=-1}$$

6.2 Aturan Rantai Kedua

Solution

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t} \\
&= (2x+y)(s)+(2y+x)(-1)+(2z)(2) \\
&= 2s^2t+s(s-t)-2(s-t)-st+4(s+2t) \\
&= 2s^2t+s^2-st-2s+2t-st+4s+8t \\
&= (2t+1)s^2-2st+2s+10t \\
\frac{\partial w}{\partial t}\Big|_{s=1,t=-1} &= (2\cdot -1+1)(1)^2-2(1)(-1)+2(1)+10(-1) \\
&= -1+2+2-10 \\
&= -7
\end{aligned}$$

6.2 Aturan Rantai Kedua

Example

Misalkan $z = x^2 y$ dengan $x = s + t$ dan $y = 1 - st$. Tentukan:

$$1)\ \frac{\partial z}{\partial s}$$
$$2)\ \frac{\partial z}{\partial t}$$

6.2 Aturan Rantai Kedua

Solution

$$
\begin{aligned}
1)\ \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \\
&= 2xy\cdot 1 + x^2\cdot(-t) \\
&= 2(s+t)(1-st) - t(s+t)^2 \\
2)\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} \\
&= 2xy\cdot 1 + x^2(-s) \\
&= 2(s+t)(1-st) - s(s+t)^2
\end{aligned}
$$

6.3 Fungsi Implisit

- Andaikan fungsi $F(x, y) = 0$ mendefinisikan y secara implisit sebagai fungsi x, misal $y = g(x)$.
- Kita akan menemukan beberapa kasus dimana kita kesulitan atau bahkan tidak mungkin menentukan fungsi g.
- Dalam kasus ini kita dapat menentukan dy/dx dengan menggunakan metode turunan implisist $(subbab\ 2.7)$.
- Namun pada subbab ini kita akan pelajari metode lain menentukan dy/dx.

6.3 Fungsi Implisit

Jika fungsi $F(x, y) = 0$ diturunkan menggunakan aturan rantai, maka diperoleh

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

Dengan demikian, dy/dx dapat diselesaikan menjadi

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

Example

Jika $x^3 + x^2 y - 10y^4 = 0$ carilah dy/dx dengan menggunakan:

1. Aturan Rantai
2. Turunan Implisit

6.3 Fungsi Implisit

Solution

1. *Dengan aturan rantai diperoleh*

$$\begin{aligned} \frac{\partial y}{\partial x} &= -\frac{\partial F/\partial x}{\partial F/\partial y} \\ &= -\frac{3x^2 + 2xy}{x^2 - 40y^3} \end{aligned}$$

2. *Dengan turunan implisit diperoleh*

$$\begin{aligned} 3x^2 + x^2\frac{dy}{dx} + 2xy - 40y^3\frac{dy}{dx} &= 0 \\ \left(x^2 - 40y^3\right)\frac{dy}{dx} &= -\left(3x^2 + 2xy\right) \\ \frac{dy}{dx} &= -\frac{3x^2 + 2xy}{x^2 - 40y^3} \end{aligned}$$

6.3 Fungsi Implisit

Jika z fungsi implisit dari x dan y yang didefinisikan oleh $F(x, y, z) = 0$, maka diferensiasi kedua ruas terhadap x dengan mempertahankan y tetap menghasilkan

$$\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

Dengan demikian, $\partial z/\partial x$ dapat diselesaikan dengan memperhatikan bahwa $\frac{\partial y}{\partial x} = 0$ menghasilkan rumus (1) . Perhitungan serupa dengan mempertahankan x tetap dan menurunkan persamaan terhadap y diperoleh rumus (2)

$$(1) \quad \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} \qquad\qquad (2) \quad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

6.3 Fungsi Implisit

Example

Carilah $\partial z/\partial x$ jika $F(x,y,z) = x^3 e^{y+z} - y \sin(x-z) = 0$ mendefinisikan z secara implisist sebagai fungsi x dan y.

Solution

$$\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{\partial F/\partial x}{\partial F/\partial z} \\ &= -\frac{3x^2 e^{y+z} - y\cos(x-z)}{x^3 e^{y+z} + y\cos(x-z)}\end{aligned}$$

6.3 Fungsi Implisit

Problem

1. *Diketahui* $x^3 + 2x^2y - y^3 = 0$. *Tentukan*

$$\frac{dy}{dx}$$

2. *Diketahui* $3x^2z + y^3 - xyz = 0$. *Tentukan*

$$a. \quad \frac{\partial z}{\partial x}$$

$$b. \quad \frac{\partial z}{\partial y}$$

6.4 Latihan 6

Problem

1. *Carilah dw/dt dengan menggunakan Aturan Rantai, nyatakan hasil akhir dan bentuk variabel t:*

$$a.\quad w = x^2y - y^2x;\quad x = \cos t,\ y = \sin t$$

$$b.\quad w = \ln\left(\frac{x}{y}\right);\ x = \tan t,\ y = \sec^2 t$$

$$c.\quad w = xy + yz + xz;\ x = t^2,\ y = 1 - t^2, z = 1 - t$$

2. *Carilah $\partial w/\partial t$ dengan menggunakan Aturan Rantai, nyatakan hasil akhir dan bentuk variabel s dan t:*

$$a.\quad w = \ln(x + y) - \ln(x - y);\quad x = te^s,\ y = e^{st}$$

$$b.\quad w = \sqrt{x^2 + y^2 + z^2};\ x = \cos st,\ y = \sin st,\ z = s^2t$$

$$c.\quad w = e^{xy+z};\ x = s + t,\ y = s - t,\ z = t^2$$

6.4 Latihan 6

Problem

3. *Jika* $z = x^2 y + z^2,\ x = \rho\cos\theta\sin\phi,\ y = \rho\sin\theta\sin\phi,$ *dan* $z = \rho\cos\phi,$*carilah*

$$\frac{\partial z}{\partial\theta}\Big|_{\rho=2,\theta=\pi,\phi=\pi/2}$$

4. *Gunakan aturan rantai fungsi implisit untuk menemukan* dy/dx *:*

   a. $ye^{-x} + 5x - 17 = 0$
   b. $x^2\cos y - y^2\sin x = 0$
   c. $x\sin y + y\cos x = 0$
   d. *Jika* $ye^{-x} + z\sin x = 0$, *Carilah* $\partial x/\partial z$

" **Terima Kasih, Semoga Bermanfaat** "