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Bifurcation and Chaos in a Discrete-Time Fractional-Order Logistic Model with Allee Effect and Proportional Harvesting

Abstract

The Allee Effect and harvesting always get a pivotal role in studying the preservation of a population. In this contex, we consider a Caputo fractional order logistic model with Allee effect and proportional harvesting. In particular, we implement the piecewise constant arguments (PWCA) method to discretize the fractional model. The dynamics of the obtained discrete-time model is then analyzed. Fixed points and their stability conditions are established. We also show the existence of saddle-node and period-doubling bifurcations in the discrete-time model. These analytical results are then confirmed by some numerical simulations via bifurcation diagram, Cobweb diagram and maximal Lyapunov exponents. The occurrance of period-doubling bifurcation route to chaos is also observed numerically. Finally, the occurence of period-doubling bifurcation is succesfully conrolled using hybrid control strategy.

Keywords: Discrete fractional-order, Logistic map, Allee effect, Harvesting, Bifurcation, Chaos

MSC Classification: 34A08 , 39A28 , 39A30 , 92D40

1 Introduction

For the last decades, the discrete-time model gets a lot of great attentiveness from researchers in mathematical modeling, not only because of its capability in describing several phenomena such as physics, biomedicine, engineering, chemistry, and population dynamics but also due to the richness of the given dynamical patterns as well as the occurrence of bifurcations and chaotic solutions which very difficult to find in their continuous counterpart [1–6]. Particularly, the discrete-time model is successfully applied in population dynamics especially in a single logistic growth modeling [7–10], the epidemic modeling [11–13], and the predator-prey interaction modeling [14–18]. Most of the models are discretized using Euler scheme [19–21] and nonstandard finite difference (NSFD) [22–24] which popular for discretization of the model with first-order derivative as the operator. Furthermore, for the model with fractional-order derivative, the popular discretization process is given by piecewise constant arguments (PWCA) which were proposed by El-Sayed et al. [25] and applied by other researchers in different biological phenomena [26–30].

In this paper, we study and justify the dynamics of a discrete-time model constructed using PWCA from a fractional-order logistic growth model involving Allee effect and harvesting. The model is given by

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)(N - m) - qN, \tag{1}$$

where $N(t)$ represents the population density at time t with $r,\ K,\ m$ and q are positive parameters represent the intrinsic growth rate, the environmental carrying capacity, the Allee effect threshold, and the harvesting rate, respectively. Notice that the Allee effect reduces the population growth rate when the population density is low (i.e., when $N < m$) as a result of several natural mechanisms such as intraspecific competition, cooperative anti-predator behavior, cooperative breeding, limitation in finding mates, and so forth. The positive growth rate occurs if the population density is in the interval $m < N < K$. For further explanation about the Allee effect, see [31–40].

To obtain the fractional-order model, we follow the similar way as in [14]. The first-order derivative at the left-hand side of model (1) is replaced with the fractional order-derivative ${}^{C}\mathcal{D}_t^{\alpha}$ which denotes the Caputo fractional derivative operator of order-α defined by

$${}^{C}\mathcal{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} ds, \tag{2}$$

where α is the order of fractional derivative with $\alpha \in (0, 1]$ and $\Gamma(\cdot)$ is the Gamma function. Furthermore, by replacing the operator along with equating the dimensions of time at the right hand-side, the following model is acquired.

$${}^{C}\mathcal{D}_t^{\alpha} N = r^{\alpha} N\left(1 - \frac{N}{K}\right)(N - m) - q^{\alpha} N. \tag{3}$$

Model (3) can be written as

$${}^{C}\mathcal{D}_t^{\alpha} N = \bar{r} N\left(1 - \frac{N}{K}\right)(N - m) - \bar{q} N, \tag{4}$$

where $\bar{r} = r^{\alpha}$ and $\bar{q} = q^{\alpha}$. Finally, by dropping $(\bar{\cdot})$, the fractional-order model for (1) is succesfully obtained as follows.

$${}^{C}\mathcal{D}_t^{\alpha} N = rN\left(1 - \frac{N}{K}\right)(N - m) - qN, \tag{5}$$

As far as we are aware, the discrete-time version of model (5) has not been introduced and studied. Hence, in this paper, we construct a such discrete-time model by implementing the PWCA method for model (5), and the dynamics of the obtained discrete-time model is then investigated. The layout of this paper is as follows. In Section 2, the model formulation is given by applying the PWCA method to get a discrete-time model. To support the analytical process, we provide some basic theoretical results in Section 3. In Section 4, some analytical results are provided such as the existence of fixed points,their local stability, the existence of saddle-node, and period-doubling bifurcations. In Section 5, we present some numerical simulations and show some interesting phenomena, such as bifurcation, Lyapunov exponent, and Cobweb diagrams which correspond to the previous theoretical results. We also present numerically a period-doubling route to chaos. A hybrid control strategy is applied to delay and eliminate the occurrence of period-doubling bifurcation and chaotic solution in Section 6. The conclusion of this work is given in Section 7.

2 Model Formulation

By applying a similar procedure as in [25, 26], we discretize model (5) with the PWCA method as follows

$$^{C}\mathcal{D}_t^{\alpha} N(t) = rN\left([t/h]\,h\right)\left(1 - \frac{N\left([t/h]\,h\right)}{K}\right)\left(N\left([t/h]\,h\right) - m\right) - qN\left([t/h]\,h\right),$$

with initial condition $N(0) = N_0$. Let $t \in [0,h)$, $t/h \in [0,1)$, then we have

$$^{C}\mathcal{D}_t^{\alpha} N(t) = rN_0\left(1 - \frac{N_0}{K}\right)(N_0 - m) - qN_0. \tag{6}$$

The solution of eq. (6) is

$$N_1 = N_0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}\left[rN_0\left(1 - \frac{N_0}{K}\right)(N_0 - m) - qN_0\right]$$

Next, let $t \in [h, 2h)$, $t/h \in [1,2)$. Thus, we obtain

$$^{C}\mathcal{D}_t^{\alpha} N(t) = rN_1\left(1 - \frac{N_1}{K}\right)(N_1 - m) - qN_1, \tag{7}$$

where its solution is given by

$$N_2 = N_1(h) + \frac{(t-h)^{\alpha}}{\Gamma(1+\alpha)}\left[rN_1\left(1 - \frac{N_1}{K}\right)(N_1 - m) - qN_1\right].$$

By proceeding the same disretization process, for $t \in [nh, (n+1)h)$, $t/h \in [n, n+1)$, we have

$$N_{n+1} = N_n(nh) + \frac{(t-nh)^\alpha}{\Gamma(1+\alpha)} \left[rN_n(nh) \left(1 - \frac{N_n(nh)}{K}\right)(N_n(nh) - m) - qN_n(nh) \right]. \tag{8}$$

For $t \to (n+1)h$, eq. (8) is reduced to a discrete fractional order logistic model with the Allee effect and proportional harvesting

$$N_{n+1} = N_n + \frac{h^\alpha}{\Gamma(1+\alpha)} N_n \left[r\left(1 - \frac{N_n}{K}\right)(N_n - m) - q \right] = f(N). \tag{9}$$

We remark that if $\alpha \to 1$ then eq. (9) is exactly the same as the Euler discretization of model (5).

3 Fundamental Concepts

To analyze the dynamical behavior such as the existence of fixed point, the local stability, and the occurrence of saddle-node and period-doubling bifurcation of the discrete-time model (9), the following definition and theorems are needed.

Definition 1 [41] Consider the following map

$$x(n+1) = f(x(n)). \tag{10}$$

A point x^* is said a fixed point of the map (10) if $f(x^*) = x^*$. If $|f'(x^*)| \neq 1$ then x^* is called a hyperbolic fixed point, and if $|f'(x^*)| = 1$ then x^* is called a nonhyperbolic fixed point.

Theorem 1 *[41] Let x^* be a hyperbolic fixed point of the map (10) where f is continuously differentiable at x^*. The following statements then hold true:*

(i) If $|f'(x^*)| < 1$, then x^* is locally asymptotically stable.
(ii) If $|f'(x^*)| > 1$, then x^* is unstable.

Theorem 2 *[41] Let x^* is a nonhyperbolic fixed point of the map (10) satisfying $f'(x^*) = 1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at x^*, then the following statements hold:*

(i) If $f''(x^*) \neq 0$, then x^* unstable (semistable).
(ii) If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then x^* unstable.
(iii) If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* locally asymptotically stable.

Definition 2 [41] The Schwarzian derivative, Sf, of a function f is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2.$$

Particularly, if $f'(x^*) = -1$ then

$$Sf(x^*) = -f'''(x^*) - \frac{3}{2}\left[f''(x^*)\right]^2.$$

Theorem 3 *[41] Let x^* is a hyperbolic fixed point of the map (10) satisfying $f'(x^*) = -1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at x^*, then the following statements hold:*

(i) If $Sf(x^*) < 0$ then x^* is locally asymptotically stable.
(ii) If $Sf(x^*) > 0$ then x^* is unstable.

Theorem 4 *(The existence of Saddle-Node Bifurcation [41]) Suppose that $x_{n+1} = f(\mu, x_n)$ is a C^2 one-parameter family of one-dimensional maps, and x^* is a fixed point with $f'(\mu, x) = 1$. Assume further that*

$$\frac{\partial f}{\partial \mu}(\mu^*, x^*) \neq 0 \text{ and } \frac{\partial^2 f}{\partial x^2}(\mu^*, x^*) \neq 0.$$

Then there exists an interval I around x^* and a C^2 map $\mu = p(x)$, where $p : I \to \mathbb{R}$ such that $p(x^*) = \mu^*$, and $f(p(x), x) = x$. Moreover, if $\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}|_{(\mu^*, x^*)} < 0$, the fixed points exist for $\mu > \mu^*$ and if $\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}|_{(\mu^*, x^*)} > 0$, the fixed points exist for $\mu < \mu^*$.

Theorem 5 *(The existence of period-doubling bifurcation [41]) Suppose that $x_{n+1} = f(\mu, x_n)$ is a C^2 one-parameter family of one-dimensional maps, and x^* is a fixed point. Assume that*

(i) $\frac{\partial f}{\partial x}(\mu^*, x^*) = -1$.
(ii) $\frac{\partial^2 f}{\partial \mu \partial x}(\mu^*, x^*) \neq 0$.

Then there is an interval I about x^* and a function $p : I \to \mathbb{R}$ such that $f_{p(x)}(x) \neq x$ but $f^2_{p(x)} = x$.

4 Analytical Results

We explore some analytical results here such as the existence of fixed points, their local stability, and the existence of some bifurcations namely saddle-node and period-doubling bifurcations. Since the map (9) is constructed by PWCA with step-size (h), the analytical process is then investigated by considering the impact of h. Some analytical results also examine the influence of the harvesting (q) on the dynamics of the given map.

4.1 The Existence of Fixed Point

Based on Definition (1), the fixed point of the map (9) is obtained by solving the following equation

$$N = N + \frac{h^\alpha}{\Gamma(1+\alpha)} N\left[r\left(1 - \frac{N}{K}\right)(N - m) - q\right]. \tag{11}$$

The solutions of eq. (11) is described as follows:

(i) The extinction of population fixed point $N_0^* = 0$ which always exists.
(ii) The non-zero fixed points $N_{1,2}^*$ which are the positive solutions of the following quadratic polynomial

$$N^2 - (m+K)N + mK + \frac{qK}{r} = 0. \tag{12}$$

The solutions of eq. (12) are

$$\begin{aligned} N_1^* &= \frac{m+K}{2} + \frac{\sqrt{(q^*-q)rK}}{r}, \\ N_2^* &= \frac{m+K}{2} - \frac{\sqrt{(q^*-q)rK}}{r}, \end{aligned} \tag{13}$$

where $q^* = \frac{(m-K)^2 r}{4K} > 0$. The existence of non-zero fixed points (13) is shown by Theorem 6.

Theorem 6 *(i) If $q > q^*$ then the non-zero fixed point of the map (9) does not exist.*
(ii) If $q = q^*$ then there exists a unique non-zero fixed point $N^* = \frac{m+K}{2}$ of the map (9).
(iii) If $q < q^*$, then there exist two non-zero fixed points, namely $N_{1,2}^*$ of the map (9).

Proof (i) It is easy to confirm that if $q > q^*$ then the solutions of eq. (12) are a pair of complex conjugate numbers.
(ii) For $q = q^*$, we have $N^* = N_1^* = N_2^* = \frac{m+K}{2}$. Hence, N^* is the only positive fixed point of the map (9).
(iii) If $q < q^*$ then $N_{1,2}^* \in \mathbb{R}$. Because $N_1^* N_2^* = mK + \frac{qK}{r} > 0$ and $N_1^* + N_2^* = m + K > 0$, then N_1^* and N_2^* are obviously positive, showing that there are two non-zero fixed points.

□

4.2 Local Stability

Now, the dynamical behavior of the map (9) around each fixed point is investigated. The following Theorems 7 to 10 is presenting to describe the local dynamics of Fixed points N_i^*, $i = 0, 1, 2$ and N^*.

Theorem 7 *Let's denote $\hat{q} = \frac{(2m^2+3mK+2K^2)r}{K}$ and $h_0 = \sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{mr+q}}$. Then the following statements hold:*

(i) if $0 < h < h_0$ then N_0^* is locally asymptotically stable,

(ii) if $h > h_0$ then N_0^* is unstable, and
(iii) if $h = h_0$ then N_0^* is nonhyperbolic fixed point. Furthermore if
(iii.a) $q < \hat{q}$ then N_0^* is locally asymptotically stable, and
(iii.b) $q > \hat{q}$ then N_0^* is unstable.

Proof By evaluating $f'(N)$ at N_0^*, we obtain

$$f'(N_0^*) = 1 - \frac{h^\alpha(mr+q)}{\Gamma(1+\alpha)} = 1 - 2\left(\frac{h}{h_0}\right)^\alpha.$$

(i) If $0 < h < h_0$ then $0 < (h/h_0)^\alpha < 1$, which implies $|f'(N_0^*)| < 1$. Based on Theorem 1, we have a locally asymptotically stable N_0^*.
(ii) If $h > h_0$ then $(h/h_0)^\alpha > 1$, so that $f'(N_0^*) < -1$. Theorem 1 states that N_0^* is an unstable fixed point.
(iii) For $h = h_0$, we have $f'(N_0^*) = -1$, i.e., N_0^* is nonhyperbolic fixed point. The Schwarzian derivative of map $f(N)$ at N_0^* is

$$\begin{aligned}
Sf(N_0^*) =& \frac{h^\alpha}{\Gamma(1+\alpha)}\left[\frac{6r}{K}\right] - \frac{3}{2}\left[\frac{h^\alpha}{\Gamma(1+\alpha)}\left[\frac{2(m+K)r}{K}\right]\right]^2\\
=& \frac{6rh^\alpha}{K\Gamma(1+\alpha)}\left[1 - \frac{h^\alpha}{K\Gamma(1+\alpha)}(m+K)^2 r\right]\\
=& \frac{12r}{(mr+q)K}\left[1 - \frac{2(m+K)^2 r}{(mr+q)K}\right].
\end{aligned}$$

If $q < \hat{q}$ then $Sf(N_0^*) < 0$, and thus N_0^* is locally asymptotically stable. On the contrary, if $q > \hat{q}$ then $Sf(N_0^*) > 0$, showing N_0^* is unstable. Thus, Theorem 7 are completely proven.

□

Theorem 8 *The non-zero fixed point N^* is semistable.*

Proof The derivative of $f(N)$ at N^* is

$$\begin{aligned}
f'(N^*) &= 1 + \frac{h^\alpha}{\Gamma(1+\alpha)}\left[-\frac{3r}{K}\left(\frac{m+K}{2}\right)^2 + \frac{2r(m+K)}{K}\left(\frac{m+K}{2}\right) - mr - q\right]\\
&= 1 + \frac{h^\alpha}{\Gamma(1+\alpha)}\left[\frac{(m+K)^2 r - 4mrK}{4K} - q\right]\\
&= 1 + \frac{h^\alpha}{\Gamma(1+\alpha)}\left[q^* - q\right].
\end{aligned}$$

N^* exists when $q = q^*$. Clearly that $f'(N^*) = 1$, and therefore N^* is a nonhyperbolic fixed point. By direct calculations, we can show that

$$f''(N^*) = \frac{h^\alpha}{\Gamma(1+\alpha)}\left[-\frac{6rN}{K} + 2\left(1 + \frac{m}{K}\right)r\right]$$

$$
\begin{aligned}
&= \frac{h^\alpha}{\Gamma(1+\alpha)}\left[-\frac{6r}{K}\frac{m+K}{2} + \frac{2(m+K)r}{K}\right] \\
&= \frac{h^\alpha}{\Gamma(1+\alpha)}\left[-\frac{3(m+K)r}{K} + \frac{2(m+K)r}{K}\right] \\
&= -\frac{(m+K)rh^\alpha}{K\Gamma(1+\alpha)} < 0.
\end{aligned}
$$

Since $f''(N^*) \neq 0$, Theorem 2 says that the fixed point N^* is semistable. □

Theorem 9 *Suppose that:*

$$
h_1 = \sqrt[\alpha]{\frac{2K\Gamma(1+\alpha)}{2(q^*-q)K + (m+K)\sqrt{(q^*-q)rK}}},
$$

$$
\hat{h} = \frac{(m+K)r + 6\sqrt{(q^*-q)K}}{\sqrt{4(q^*-q)rK + 2r(m+K)\sqrt{(q^*-q)rK}}}.
$$

The local stability of N_1^* is described as follows.

(i) If $0 < h < h_1$ then N_1^* is locally asymptotically stable.
(ii) If $h > h_1$ then N_1^* is unstable.
(iii) If $h = h_1$ and
(iii.a) If $\hat{h} > 1$ then N_1^* is locally asymptotically stable.
(iii.b) If $\hat{h} < 1$ then N_1^* is unstable.

Proof It is obvious to show that

$$
\begin{aligned}
f'(N_1^*) &= 1 + \tfrac{h^\alpha}{\Gamma(1+\alpha)}\Big[-\Big(\tfrac{3}{4K}(m+K)^2 r + 3(q^*-q) + \tfrac{3}{K}(m+K)\sqrt{(q^*-q)rK}\Big) \\
&\quad + \tfrac{1}{K}(m+K)^2 r + \tfrac{2}{K}(m+K)\sqrt{(q^*-q)rK} - (mr+q)\Big] \\
&= 1 - \tfrac{h^\alpha}{K\Gamma(1+\alpha)}\Big[2(q^*-q)K + (m+K)\sqrt{(q^*-q)rK}\Big] \\
&= 1 - \tfrac{h^\alpha}{K\Gamma(1+\alpha)}\Big[\tfrac{2K\Gamma(1+\alpha)}{h_1^\alpha}\Big] \\
&= 1 - 2\Big(\tfrac{h}{h_1}\Big)^\alpha.
\end{aligned}
$$

Hence, we have the following observations:

(i) For $0 < h < h_1$, we have $|f'(N_1^*)| < 1$. According to Theorem 1, the non-zero fixed point N_1^* is locally asymptotically stable.
(ii) If $h > h_1$ then we get $f'(N_1^*) < -1$. Thus, N_1^* is unstable fixed point (see Theorem 1).
(iii) Clearly that $f'(N_1^*) = -1$ whenever $h = h_1$, which shows that N_1^* is non-hyperbolic fixed point. The Schwarzian derivative of $f(N)$ at N_1^* is given by

$$
\begin{aligned}
Sf(N_1^*) &= \tfrac{h^\alpha}{\Gamma(1+\alpha)}\Big[\tfrac{6r^2}{K}\Big] - \tfrac{3}{2}\Big[-\tfrac{h^\alpha}{\Gamma(1+\alpha)}\Big[\tfrac{(m+K)r}{K} + \tfrac{6}{K}\sqrt{(q^*-q)K}\Big]\Big]^2 \\
&= \tfrac{rh^\alpha}{K\Gamma(1+\alpha)}\Big[6r - \tfrac{3}{2K}\Big[\tfrac{h^\alpha}{\Gamma(1+\alpha)}\Big]\Big[(m+K)r + 6\sqrt{(q^*-q)K}\Big]^2\Big].
\end{aligned}
$$

We can easily check that if $\hat{h} > 1$ then $Sf(N_1^*) < 1$ and if $\hat{h} < 1$ then $Sf(N_1^*) > 1$. Therefore, the stability of the nonhyperbolic fixed point is explained. Finally, all of the stability conditions of fixed point N_1^* are completely determined.

□

Theorem 10 *The non-zero fixed point* $N_2^* = \frac{m+K}{2} - \frac{\sqrt{(q^*-q)rK}}{r}$ *is always unstable.*

Proof To investigate the stability of N_2^*, we evaluate $f'(N)$ at N_2^*:

$$f'(N_2^*) = 1 + \frac{(q^* - q)h^\alpha}{\Gamma(1+\alpha)} \left[\frac{(m+K)\sqrt{r}}{\sqrt{(q^*-q)K}} - 2 \right].$$

By simple algebraic manipulations, we can show that $\frac{(m+K)\sqrt{r}}{\sqrt{(q^*-q)K}} > 2$. Thus, $f'(N_2^*)$ is always a positive constant, which means N_2^* is always an unstable fixed point. □

4.3 Bifurcation Analysis

From the previous analysis, we have a non-hyperbolic fixed point N^* when $q = q^*$, indicating the possibility of the occurrence of saddle-node bifurcation. Moreover, the occurrence of period-doubling bifurcation is also indicated around the non-hyperbolic fixed point N_1^* when $h = h_1$. Thus, in this section, we study the existence of saddle-node and period-doubling bifurcations.

Theorem 11 *The non-zero fixed point* N^* *undergoes a saddle-node bifurcation when* q *crosses the critical values* $q^* = \frac{(m-K)^2 r}{4K}$.

Proof It was shown previously that N^* does not exist if $q > q^*$. When $q = q^*$ we have a semistable fixed point N^*; and if $q < q^*$, then there exists two non-zero fixed points. By straightforward calculations, we have $\frac{\partial f(N^*)}{\partial N} = 1$, $\frac{\partial f(N^*)}{\partial q} = -\frac{h^\alpha}{\Gamma(1+\alpha)} \frac{m+K}{2} < 0$, and $\frac{\partial^2 f(N^*)}{\partial N^2} = -\frac{(m+K)rh^\alpha}{K\Gamma(1+\alpha)} < 0$. Thus, according to Theorem 4, the fixed point N^* undergoes a saddle-node bifurcation when q crosses the critical values $q^* = \frac{(m-K)^2 r}{4K}$. Moreover, the fixed points exist when $q \leq q^*$ because $\frac{\partial f(N^*)}{\partial q} \frac{\partial^2 f(N^*)}{\partial N^2} > 0$. □

Theorem 12 *The non-zero fixed point* N_1^* *undergoes a period-doubling bifurcation when* h *crosses the critical value* $h_1 = \sqrt[\alpha]{\frac{2K\Gamma(1+\alpha)}{2(q^*-q)K+(m+K)\sqrt{(q^*-q)rK}}}$.

Proof From the proof of Theorem 9 we have that if $h = h_1$ then $\frac{\partial f(N_1^*)}{\partial N} = -1$. By performing some algebraic calculations, we also have

$$\begin{aligned}\frac{\partial^2 f(N_1^*)}{\partial h \partial N} =& \frac{\alpha h^{\alpha-1}}{\Gamma(1+\alpha)}\left[r\left(\frac{m+K}{2}+\frac{\sqrt{(q^*-q)rK}}{r}\right)\right. \\ & \left.\left(1-\frac{2}{K}\left(\frac{m+K}{2}+\frac{\sqrt{(q^*-q)rK}}{r}\right)+\frac{m}{K}\right)\right] \\ =& -\frac{r\alpha h^{\alpha-1}}{\Gamma(1+\alpha)}\left(\frac{m+K}{2}+\frac{\sqrt{(q^*-q)rK}}{r}\right)\left(\frac{2}{K}\frac{\sqrt{(q^*-q)rK}}{r}\right).\end{aligned} \tag{14}$$

N_1^* exists if $q \neq q^*$ and thus we have $\frac{\partial^2 f(N_1^*)}{\partial h \partial N} \neq 0$. According to Theorem 5, there appears a solution of period-2 when h passes through h_1. Hence, the occurrence of period-doubling bifurcation in the map (9) is completely proven. □

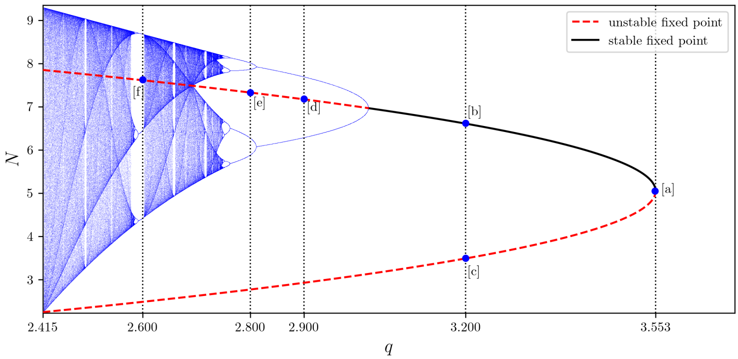
Theorem 12 states that the period-doubling bifurcation in the map (9) can be achieved by varying the step size h. However such bifurcation can also be realized by setting a fixed value of h and other parameters while varying a certain parameter. In the following Section, we give an example of period-doubling bifurcation which is driven by the constant of harvesting (q).

5 Numerical Results

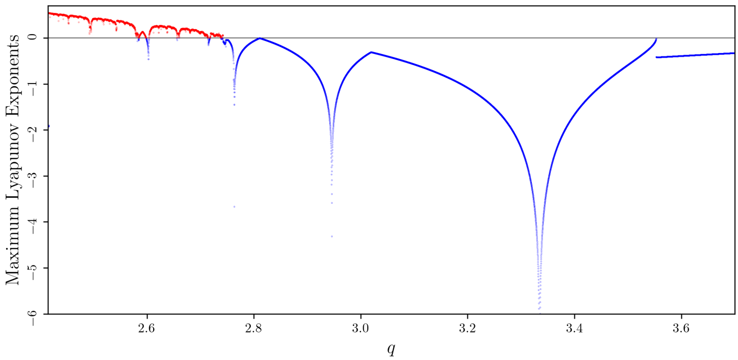
In this section, we present some numerical simulations of the map (9) to support the previous analytical findings. Due to the field data limitation, we use hypothetical parameter values for the numerical simulations. We begin with a simulation using the following parameter values:

$$r = 1.45, \quad K = 10, \quad m = 0.1, \quad \text{and } \alpha = 0.8. \tag{15}$$

According to Theorem 6, map (9) with parameter set (15) has critical value $q^* \approx 3.5527$ such that map (9) does not have non-zero fixed point if $q > q^*$. When $q = q^*$, map (9) has a unique non-zero fixed point $N^* = 5.05$ which is a semistable fixed point, see Theorem 8. Furthermore, if $q < q^*$, then there are two non-zero fixed points, namely N_1^* and N_2^*. By taking $h = 0.4$ and using Theorem 9, we can show that N_1^* is asymptotically stable if $q_1 = 3.0191 \lesssim q < q^*$. On the other hand, Theorem 10 states that N_2^* is always unstable. Since we take $h = 0.4$, Theorem 12 states that the fixed point N_1^* undergoes a period-doubling bifurcation when q crosses q_1 from the right. To see these dynamical behaviors, we plot in fig. 1a the bifurcation diagram of the map (9) with parameter set (15) and $h = 0.4$ for $2.415 \leq q \leq 3.7$. Clearly that this bifurcation diagram fits perfectly with the results of our previous analysis. Indeed, fig. 1a shows that N^* (labelled as [a]) is semistable, see also the Cobweb diagram shown in fig. 2a. As the value of q decreases from q^*, the non-zero fixed point is split into two non-zero fixed points where one of them is stable in the specified interval of q, while the other fixed point is unstable. Such stability properties can also be seen from the Cobweb diagrams in fig. 2b



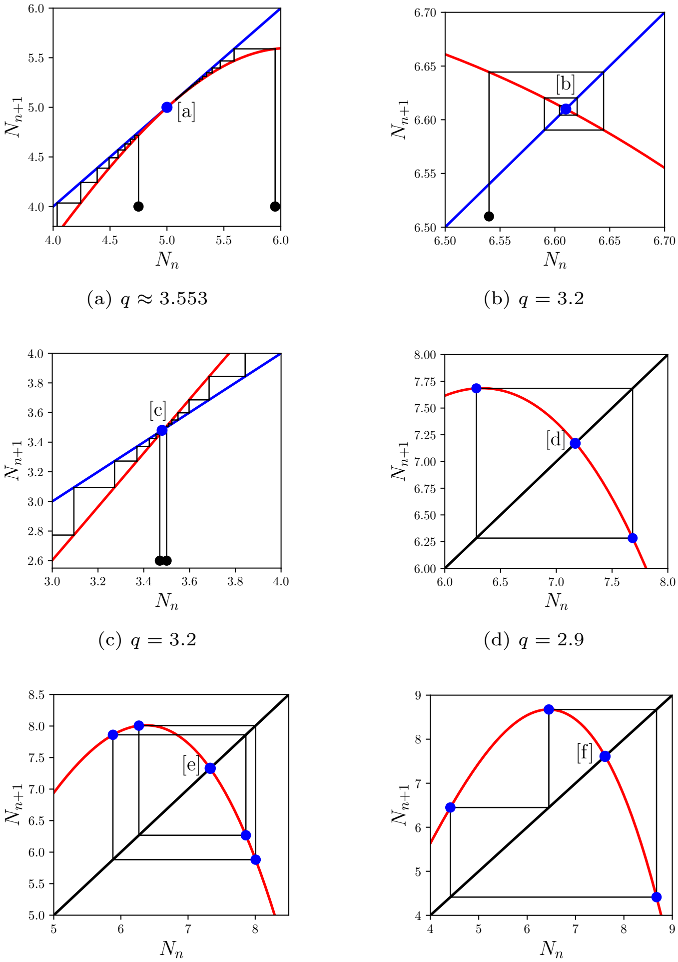
(a) Bifurcation diagram



(b) Maximum Lyapunov exponents

Fig. 1: (a) Bifurcation diagram of the map (9) with parameter set (15), $h = 0.4$, and $2.415 \leq q \leq 3.7$ and (b) the corresponding maximum Lyapunov exponents.

and fig. 2c, which correspond to point [b] and [c] in fig. 1a, respectively. We also observe numerically the appearance of a period-doubling route to chaos (flip bifurcation) as q decreases. If we further decrease the value of q, then there appears a stable solution of period-2 when q passes through q_1. The appearance of a stable period-doubling solution, as well as a solution of period-3, are clearly seen in fig. 1a (see e.g. point [d], [e] and [f], respectively and their corresponding diagram Cobweb in fig. 2d, 2e and 2f). The appearance of the period-3 solution indicates that our system exhibits chaotic dynamics [42]. The existence of chaotic dynamics can also be determined from the Lyapunov exponent. A system exhibits chaotic dynamics if it has positive maximum Lyapunov exponents. The maximum Lyapunov exponents which correspond to fig. 1a is depicted in fig. 1b. It is clearly seen that our system has positive maximum Lyapunov exponents, showing the existence of chaotic dynamics in the map (9) which is controlled by the constant of harvesting (q).



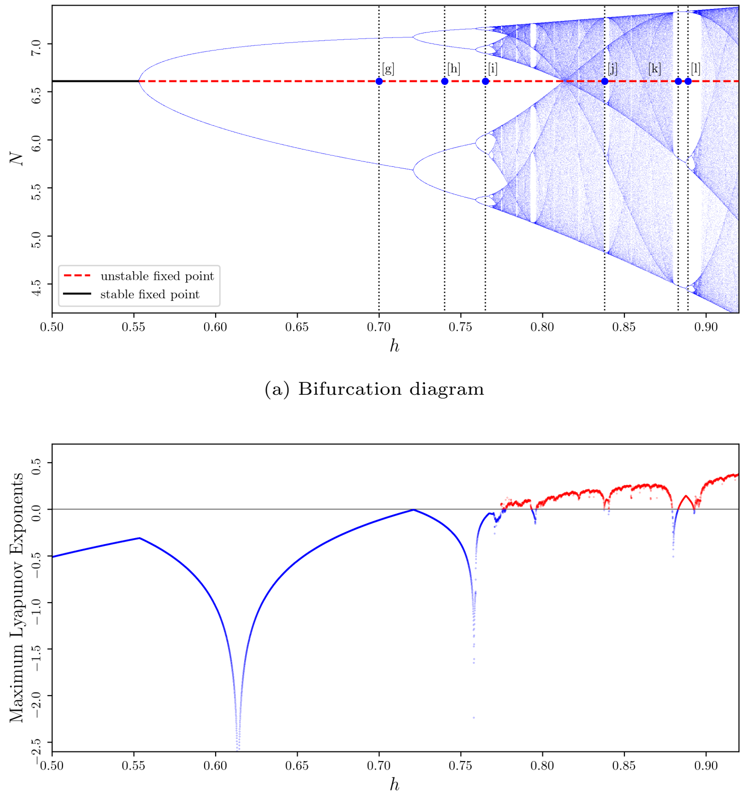
(a) $q \approx 3.553$ (b) $q = 3.2$

(c) $q = 3.2$ (d) $q = 2.9$

(e) $q = 2.8$ (f) $q = 2.6$

Fig. 2: Cobweb diagrams of the map (9) with parameter set (15) and $h = 0.4$.

To describe the existence of period-doubling bifurcation driven by the step-size h numerically, we perform simulations using the parameter set (15), $q = 3.2$, and $0.5 \leq h \leq 0.985$. Map (9) with these parameter values has two non-zero fixed points, namely $N_1^* \approx 6.61$ and $N_2^* \approx 3.49$. N_2^* is unstable while N_1^* is stable if $0 < h < h_1 \approx 0.553$. N_1^* losses its stability via period-doubling bifurcation when h crosses h_1. These dynamics are clearly seen in the bifurcation diagram, see fig. 3a. Increasing the value of h may destroy the stability of N_1^* and the system is convergent to a stable period-2 solution. Further increasing the value of h leads to a stable period-4 cycle, and so on. To give a more detailed view, in fig. 4 we plot Cobweb diagrams which correspond

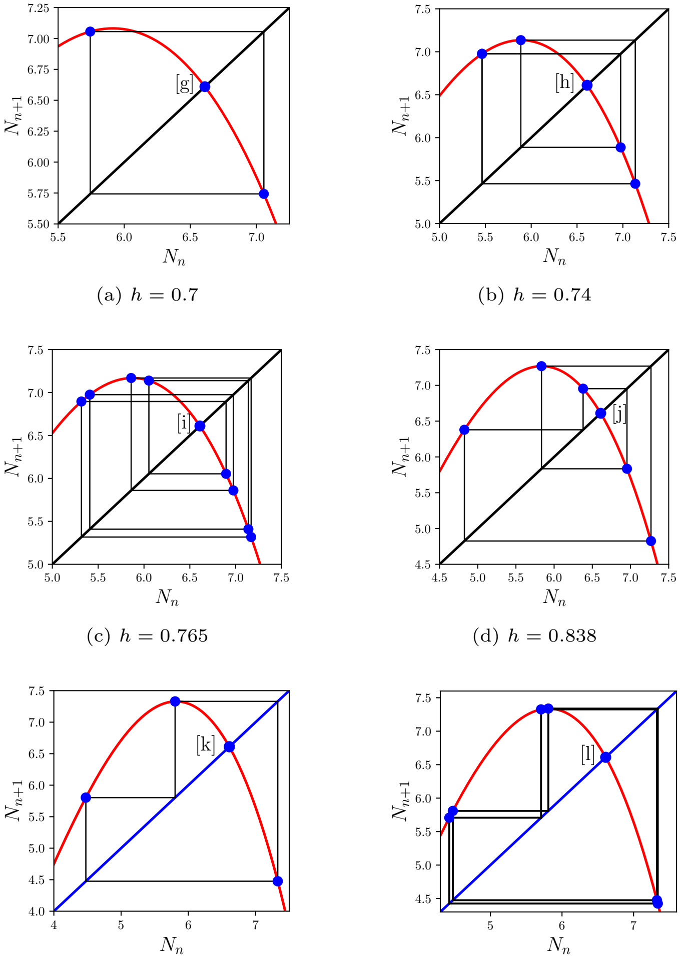


(a) Bifurcation diagram

(b) Maximum Lyapunov exponents

Fig. 3: (a) Bifurcation diagram of the map (9) with parameter set (15), $q = 3.2$, and $0 \leq h \leq 0.92$ and (b) the corresponding maximum Lyapunov exponents.

to some solutions around the fixed points labeled as [g - l] in fig. 3a. When $h = 0.7$, we have a stable period-2 cycle near the non-zero fixed point [g], see fig. 4a. Each of the two solutions splits into two solutions respectively, and become a stable period-4 solution around fixed point [h] when $h = 0.74$ (fig. 4b); and consecutively for $h = 0.765$ we have a stable period-8 cycle near fixed point [i], see fig. 4c. Moreover, at $h = 0.838, 0.883, 0.889$ we have respectively a stable period-5 cycle around fixed point [j], a stable period-3 cycle around fixed point [k], and a stable-period-6 cycle around fixed point [l], see their Cobweb diagrams in fig. 4d-4f. Hence, the step size h is an important parameter that significantly affects the dynamics of the map (9). In this case, the map (9) exhibits a period-doubling bifurcation route to chaos driven by parameter h. Furthermore, the appearance of positive maximum Lyapunov exponents depicted in fig. 3b which corresponds to the bifurcation diagram in fig. 3a clearly shows the existence of chaotic behavior in the system.



(a) $h = 0.7$ (b) $h = 0.74$

(c) $h = 0.765$ (d) $h = 0.838$

(e) $h = 0.883$ (f) $h = 0.889$

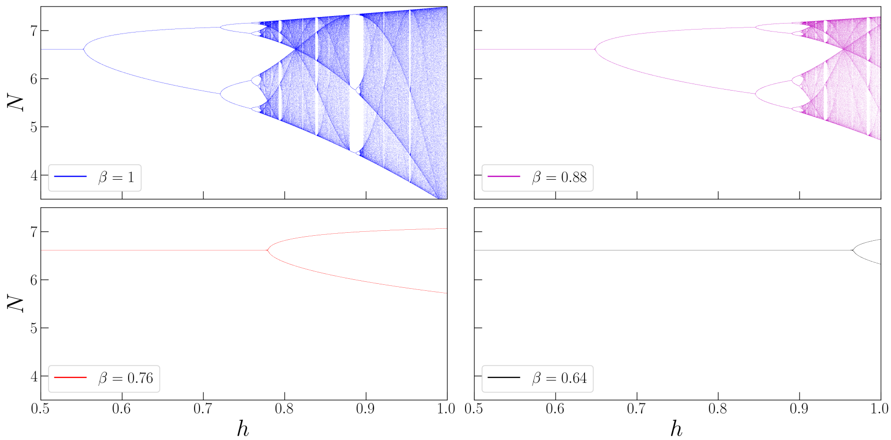
Fig. 4: Cobweb diagrams of the map (9) with parameter set (15) and $q = 3.2$.

6 Hybrid Control Strategy

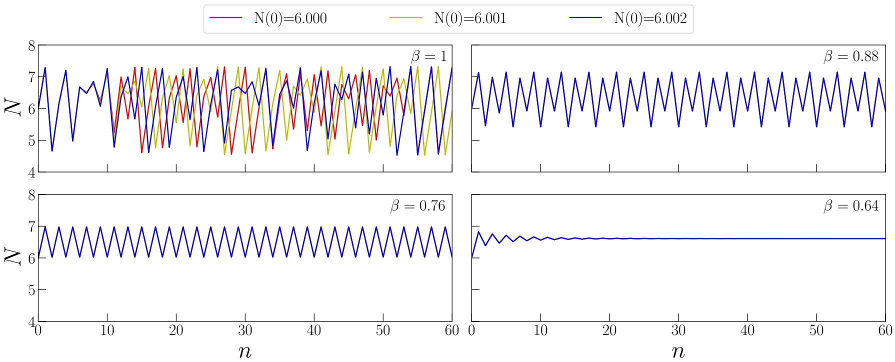
In this section, a method namely the hybrid control strategy is presented. This method is a combination of state feedback and parameter perturbation which is used for controlling bifurcation in a discrete system [43–46]. We first define a map (9) as follows.

$$N_{n+1} = f(N_n, \zeta), \tag{16}$$

where $N \in \mathbb{R}$ is the population density and, $F(N_n, \zeta)$ is the right hand side of map (9) with bifurcation parameter $\zeta \in \mathbb{R}$. It can be revisited from analytical and numerical results that when h and q are varies in some range, the map (9) passes through a series of period-doubling bifurcations where the route to



(a) Bifurcation diagram



(b) Solutions for $h = 0.877$

Fig. 5: Bifurcation diagrams of controlled map (17)

chaos. By obeying state feedback and parameter perturbation to the map (9), we obtain the control map as follows.

$$N_{n+1} = \beta f(N_n, \zeta) + (1 - \beta)N_n = F(N, \beta), \tag{17}$$

where $\beta \in [0, 1]$ denotes the external control parameter for map (17). We can easily show that the map (9) and (17) have similar fixed points. From Theorem 12, N_1^* is the fixed point which undergoes a period-doubling bifurcation. Particularly, From Theorem 1 in [46], the m-periodic orbit of control map (17) is also similar with the original map (9). Now, we will show that by setting β and varying h, the occurrence of period-doubling bifurcation can be delayed or even eliminated. From the control map (17) we have $F'(N_1^*) = 1 - 2\beta \left(\frac{h}{h_1}\right)^{\alpha}$ and $\frac{\partial^2 F(N_1^*)}{\partial h \partial N} = \frac{\partial^2 f(N_1^*)}{\partial h \partial N} < 0$. According to Theorem 5, the control map (17) also undergoes period-doubling bifurcation for the similar fixed point with map (9). The difference lies on the bifurcation point where for the map (9) is $h = h_1$ while the control map (17) is $h = \frac{h_1}{\sqrt[\alpha]{\beta}}$. This means if β decreases then the bifurcation point increase which means the series of periodic solutions are delayed. For example, by setting the parameter values as in eq. (15) and $\beta = 0.64, 0.76, 0.88, 1$, the occurrence of bifurcation delayed and period-3 solutions disappears. See Figure 5a. We also check the chaotic solution near the period-3 solution. For $h = 0.887$, three quite close initial conditions $N(0) = 6, 6.001, 6.002$ is given and portray the solutions in Figure 5b. The chaotic interval which occurs for $\beta = 1$, becomes a periodic solution for $\beta = 0.76, 0.88$, and finally converge to N_1^* when $\beta = 0.64$.

7 Conclusion

A discrete fractional order logistic model with the Allee effect and proportional harvesting has been constructed and investigated dynamically. The discrete-time model is derived by applying the PWCA method to the Caputo fractional order modified logistic model. It was shown analytically that the obtained discrete-time model exhibits a saddle-node bifurcation as well as period-doubling bifurcation. The key parameter in such bifurcations is the constant of harvesting (q) or the step size (h). Numerical simulations with varying parameters q and h confirm our analytical results. Furthermore, the presented numerical results also showed the existence of period-doubling route chaos. We then constuct the control based on the hybrid control strategy method. It is shown that the occurrence of period-doubling can be delayed. The occurrence of the chaotic solution is also successfully eliminated when the control parameter is decreased.

Declarations

Author Contributions. All authors contributed equally in this article. They read and approved the final manuscript.

Data availability. Not applicable.

Conflict of interest. All authors declare that they have no conflicts of interest.

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6. Update the review literature with recent relevant work in the field of numerical methods for handling Time-fractional problems, for instance, https://doi.org/10.1007/s40314-022-02096-7 and https://doi.org/10.3934/math.2022830

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Manuscript Number: IJDY-D-22-00714

This paper considers a Caputo fractional-order logistic model with the Allee effect and proportional harvesting. They implement the piecewise constant arguments method to discretize the fractional model. Theoretically, the existence and stability conditions of the fixed points of the discrete model are given. Furthermore, the existence of saddle-node bifurcation and period-doubling bifurcation is proven. Finally, these analytical results are then confirmed by some numerical simulations via bifurcation diagrams, Cobweb diagrams, and maximal Lyapunov exponents. Some questions and comments need to be considered by the authors:

1. This paper studies a fractional-order model. Has the author considered the comparison between this model and the integer-order model?

2. The factors considered in the model established in this paper are fractional order, the Allee effect, and Harvesting. However, the author did not reflect on the impact of fractional order and the Allee effect on the model dynamics in the numerical simulation. As far as I know, these two factors will also bring about complex dynamic phenomena.

3. There are some minor mistakes in the manuscript that need the author's attention, for example, "conrolled" in the abstract, model (5) should not use commas, and the title and Y-axis of Fig.1 should be unified, etc.

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Dear Mr. Panigoro,

We acknowledge, with thanks, receipt of the revised version of your manuscript, "Bifurcation and Chaos in a Discrete-Time Fractional-Order Logistic Model with Allee Effect and Proportional Harvesting", submitted to International Journal of Dynamics and Control

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Bifurcation and Chaos in a Discrete-Time Fractional-Order Logistic Model with Allee Effect and Proportional Harvesting

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Abstract:	The Allee Effect and harvesting always get a pivotal role in studying the preservation of a population. In this contex, we consider a Caputo fractional order logistic model with Allee effect and proportional harvesting. In particular, we implement the piecewise constant arguments (PWCA) method to discretize the fractional model. The dynamics of the obtained discrete-time model is then analyzed. Fixed points and their stability conditions are established. We also show the existence of saddle-node and period-doubling bifurcations in the discrete-time model. These analytical results are then confirmed by some numerical simulations via bifurcation diagram, Cobweb diagram and maximal Lyapunov exponents. The occurrance of period-doubling bifurcation route to chaos is also observed numerically. Finally, the occurence of period-doubling bifurcation is succesfully conrolled using hybrid control strategy.
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Author Comments:	Dear Prof. Dr. Jian-Qiao Sun, University of California, Merced, CA, USA Editor-in-Chief of International Journal of Dynamics and Control Updates: We have revised our manuscript based on the suggestion and advice from both two reviewers. See the blue texts in the manuscript and the detail in Author's Response to Reviewers' Comments PDF document. In this manuscript, we proposed a discrete-time model involving Allee effect and propotional harvesting. The model is obtained from the fractional-order model which discretized using piecewise constant argument. Although the proposed model looks simple, some novelty are successfully shown. The following are the key contribution: 1. The discrete-time logistic model with Allee effect and propotional harvesting are naturally related to real phenomena. This model has never been studied before. 2. Complex dynamics are successfully shown namely local stability; existence of saddle-node and period-doubling bifurcation; and existence of period-3 solution which route to chaos.

	3. Numerical simulations are provided to support and explore more the dynamical behavior such as cobweb diagrams, bifurcation diagrams,Maximum Lyapunov exponent diagram, and times-series. 4. The hybrid control strategy is given to show the way to control the occurrence of bifurcation and chaotic solution. These results are expected to be considered as the advantages of this manuscript so that it meets the criteria of International Journal of Dynamics and Control. Your Sincerely, Hasan S. Panigoro Universitas Negeri Gorontalo, Indonesia
Response to Reviewers:	Dear Prof. Jian-Qiao Sun, Editor in Chief Of International Journal of Dynamics and Control We have revised our manuscript based on the suggestion and advice from both two reviewers. To facilitate and to make it easier to find the parts we have revised, we have colored in blue the modified parts and provided the details below. Reviewer 1 This paper considers a Caputo fractional-order logistic model with the Allee effect and proportional harvesting. They implement the piecewise constant arguments method to discretize the fractional model. Theoretically, the existence and stability conditions of the fixed points of the discrete model are given. Furthermore, the existence of saddle-node bifurcation and period-doubling bifurcation is proven. Finally, these analytical results are then confirmed by some numerical simulations via bifurcation diagrams, Cobweb diagrams, and maximal Lyapunov exponents. Some questions and comments need to be considered by the authors: 1. This paper studies a fractional-order model. Has the author considered the comparison between this model and the integer-order model? Answer : In this paper, we only study the discrete-time model constructed from its fractional-order model. We didn’t compare the model with those two operators by considering the fractional-order model in a one-dimensional model has poor dynamics since $\|\arg(\lambda)\|$ of the equilibrium point only has three conditions i.e., positive, negative, and zero. Those dynamics are qualitatively the same as the first-order model, and the difference only lies in the convergence rate. In another hand, the discrete-time model proposed in this paper has rich dynamics such as the existence of period-doubling bifurcation and the occurrence of chaotic dynamics. We add this explanation on page 3 lines 47-58. 2. The factors considered in the model established in this paper are fractional order, the Allee effect, and Harvesting. However, the author did not reflect on the impact of fractional order and the Allee effect on the model dynamics in the numerical simulation. As far as I know, these two factors will also bring about complex dynamic phenomena. Answer: We have added the influence of the Allee effect and the order-α to our manuscript. See pages 15-18 lines 269-300. 3. There are some minor mistakes in the manuscript that need the author's attention, for example, "conrolled" in the abstract, model (5) should not use commas, and the title and Y-axis of Fig.1 should be unified, etc. Answer: We have modified and revised some typos. For the figures, we use the sub-caption rather than title and hence the position for each sub-caption adapted to LaTeX templates. Reviewer 2 Herein, the author(s) consider a Caputo fractional order logistic model with Allee effect and proportional harvesting. Especially, they implement the piecewise constant arguments method to discretize the fractional model. The dynamics of the obtained

discrete-time model is then analyzed. Fixed points and their stability conditions are established. Also they show the existence of saddle-node and period doubling bifurcations in the discrete-time model. These analytical results are then confirmed by some numerical simulations via bifurcation diagram, Cobweb diagram and maximal Lyapunov exponents. The article needs a careful revision before the possibility of its consideration for publication in IJDY, the following comments should be addressed:

1. Extend the introductory section to cover the pros/cons of the work.

Answer: We extend the introduction on page 3 lines 47-58 to show some explanation about the based model such as first-order, fractional-order, and discrete-time models, and the reason prefers to study the discrete-time model.

2. A table of notation is required before Section 2.

Answer: We have added table 1 on page 2.

3. The target of each theorem should be reported before the theorem.

Answer: We have added some explanations for local dynamics for every fixed point on page 7 lines 135-141. We also explore more the description of bifurcation analysis on page 10 lines 179-189.

4. Mention the specs of the machine used to depict all figures as well as the platform used to generate these figures.

Answer: We have mentioned the specification of the PC along with the software for numerical simulations on page 11 lines 208-209.

5. Extend the conclusion section to cover the future extensions in this direction.

Answer: We have extended the conclusion based on the reviewers' suggestions. See the blue texts on pages 19-20 lines 322-340/

6. Update the review literature with recent relevant work in the field of numerical methods for handling Time-fractional problems, for instance, https://doi.org/10.1007/s40314-022-02096-7 and https://doi.org/10.3934/math.2022830

Answer: We have updated the references to cover relevant works to this model.

All of the reviewers' suggestion and advice have been revised and modified. We hope the revised version of this manuscript meets the requirement of the International Journal of Dynamics and Control.

Your Sincerely,
Authors

Bifurcation and Chaos in a Discrete-Time Fractional-Order Logistic Model with Allee Effect and Proportional Harvesting

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Abstract

The Allee Effect and harvesting always get a pivotal role in studying the preservation of a population. In this contex, we consider a Caputo fractional order logistic model with Allee effect and proportional harvesting. In particular, we implement the piecewise constant arguments (PWCA) method to discretize the fractional model. The dynamics of the obtained discrete-time model is then analyzed. Fixed points and their stability conditions are established. We also show the existence of saddle-node and period-doubling bifurcations in the discrete-time model. These analytical results are then confirmed by some numerical simulations via bifurcation diagram, Cobweb diagram and maximal Lyapunov exponents. The occurrance of period-doubling bifurcation route to chaos is also observed numerically. Finally, the occurence of period-doubling bifurcation is succesfully conrolled using hybrid control strategy.

Keywords: Discrete fractional-order, Logistic map, Allee effect, Harvesting, Bifurcation, Chaos

MSC Classification: 34A08 , 39A28 , 39A30 , 92D40

Declarations

Author Contributions. All authors contributed equally in this article. They read and approved the final manuscript.

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Data availability. Not applicable.

Conflict of interest. All authors declare that they have no conflicts of interest.

Bifurcation and Chaos in a Discrete-Time Fractional-Order Logistic Model with Allee Effect and Proportional Harvesting

Abstract

The Allee Effect and harvesting always get a pivotal role in studying the preservation of a population. In this contex, we consider a Caputo fractional order logistic model with the Allee effect and proportional harvesting. In particular, we implement the piecewise constant arguments (PWCA) method to discretize the fractional model. The dynamics of the obtained discrete-time model are then analyzed. Fixed points and their stability conditions are established. We also show the existence of saddle-node and period-doubling bifurcations in the discrete-time model. These analytical results are then confirmed by some numerical simulations via bifurcation, Cobweb, and maximal Lyapunov exponents diagrams. The occurrence of period-doubling bifurcation route to chaos is also observed numerically. Finally, the occurrence of period-doubling bifurcation is successfully controlled using a hybrid control strategy.

Keywords: Discrete-time fractional-order, Logistic map, Allee effect, Harvesting, Bifurcation, Chaos

MSC Classification: 34A08 , 39A28 , 39A30 , 92D40

1 Introduction

For the last decades, the discrete-time model gets a lot of great attentiveness from researchers in mathematical modeling, not only because of its capability in describing several phenomena such as physics, biomedicine, engineering, chemistry, and population dynamics but also due to the richness of the given dynamical patterns as well as the occurrence of bifurcations and chaotic solutions which very difficult to find in their continuous counterpart [1–6]. Particularly, the discrete-time model is successfully applied in population dynamics, especially in a single logistic growth modeling [7–10], the epidemic modeling [11–13], and the predator-prey interaction modeling [14–18]. Most of the models are discretized using Euler scheme [19–21] and nonstandard finite difference (NSFD) [22–24] which is popular for the discretization of the model with first-order derivative as the operator. Furthermore, for the model with fractional-order derivative, we have some numerical schemes to approximate the exact solution such in [25–28]. We also have the popular discretization process is given by piecewise constant arguments (PWCA) which were proposed by El-Sayed et al. [29] and applied by other researchers in different biological phenomena [30–34].

In this paper, we study and justify the dynamics of a discrete-time model constructed using PWCA from a fractional-order logistic growth model involving the Allee effect and harvesting. The model is given by

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)(N - m) - qN, \tag{1}$$

where $N(t)$ represents the population density at time t and all parameters are positive numbers with biological interpretations are given in Table 1.

Table 1: Biological Interpretation for each Parameter

Parameters	Biological Interpretation
r	the intrinsic growth rate
K	the environmental carrying capacity
m	the Allee effect threshold
q	the harvesting rate

Notice that the Allee effect reduces the population growth rate when the population density is low (i.e., when $N < m$) as a result of several natural mechanisms such as intraspecific competition, cooperative anti-predator behavior, cooperative breeding, limitation in finding mates, and so forth. The positive growth rate occurs if the population density is in the interval $m < N < K$. For further explanation about the Allee effect, see [35–44].

To obtain the fractional-order model, we follow a similar way as in [14]. The first-order derivative at the left-hand side of model (1) is replaced with the fractional order-derivative ${}^{C}\mathcal{D}_t^{\alpha}$ which denotes the Caputo fractional derivative operator of order-α defined by

$${}^{C}\mathcal{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} ds, \tag{2}$$

where α is the order of fractional derivative with $\alpha \in (0, 1]$ and $\Gamma(\cdot)$ is the Gamma function. Furthermore, by replacing the operator with equating the dimensions of time at the right-hand side, the following model is acquired.

$${}^{C}\mathcal{D}_t^{\alpha} N = r^{\alpha} N\left(1 - \frac{N}{K}\right)(N - m) - q^{\alpha} N. \tag{3}$$

Model (3) can be written as

$$ {}^{C}\mathcal{D}_t^{\alpha} N = \bar{r} N \left(1 - \frac{N}{K}\right)(N - m) - \bar{q} N, \tag{4} $$

where $\bar{r} = r^{\alpha}$ and $\bar{q} = q^{\alpha}$. Finally, by dropping $(\bar{\cdot})$, the fractional-order model for (1) is successfully obtained as follows.

$$ {}^{C}\mathcal{D}_t^{\alpha} N = r N \left(1 - \frac{N}{K}\right)(N - m) - q N. \tag{5} $$

As far as we are aware, both the fractional-order model and the discrete-time version of eq. (5) have not been introduced and studied. Especially for fractional-order model (5), since the stability properties of equilibrium point refers to Matignon condition [45], the dynamics of the one dimensional first-order and the fractional-order models are qualitatively the same because the $|\arg(\lambda)|$ of equilibrium point always in the real line. In the other hand, although in the one-dimensional model, the discrete-time model has more possible complex phenomena such as period-doubling bifurcation and chaotic behaviors which do not exist in its continuous ones. This means the one-dimensional continuous model has poor dynamics than the discrete-time model. Hence, for this case, studying the discrete-time model is more interesting and attractive. In this paper, we construct a discrete-time model by implementing the PWCA method for the model (5), and the dynamics of the obtained discrete-time model are then investigated. The layout of this paper is as follows. In Section 2, the model formulation is given by applying the PWCA method to get a discrete-time model. To support the analytical process, we provide some basic theoretical results in Section 3. In Section 4, some analytical results are provided such as the existence of fixed points, their local stability, the existence of saddle-node, and period-doubling bifurcations. In Section 5, we present some numerical simulations and show some interesting phenomena, such as bifurcation, Lyapunov exponent, and Cobweb diagrams which correspond to the previous theoretical results. We also present numerically a period-doubling route to chaos. A hybrid control strategy is applied to delay and eliminate the occurrence of period-doubling bifurcation and chaotic solution in Section 6. The conclusion of this work is given in Section 7.

2 Model Formulation

By applying a similar procedure as in [29, 30], we discretize model (5) with the PWCA method as follows

$$ {}^{C}\mathcal{D}_t^{\alpha} N(t) = r N\left(\left[t/h\right] h\right)\left(1 - \frac{N\left(\left[t/h\right] h\right)}{K}\right)\left(N\left(\left[t/h\right] h\right) - m\right) - q N\left(\left[t/h\right] h\right), $$

with initial condition $N(0) = N_0$. Let $t \in [0, h)$, $t/h \in [0, 1)$, then we have

$$^{C}\mathcal{D}_t^{\alpha} N(t) = rN_0 \left(1 - \frac{N_0}{K}\right)(N_0 - m) - qN_0. \tag{6}$$

The solution of eq. (6) is

$$N_1 = N_0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} \left[rN_0 \left(1 - \frac{N_0}{K}\right)(N_0 - m) - qN_0 \right]$$

Next, let $t \in [h, 2h)$, $t/h \in [1, 2)$. Thus, we obtain

$$^{C}\mathcal{D}_t^{\alpha} N(t) = rN_1 \left(1 - \frac{N_1}{K}\right)(N_1 - m) - qN_1, \tag{7}$$

where its solution is given by

$$N_2 = N_1(h) + \frac{(t-h)^{\alpha}}{\Gamma(1+\alpha)} \left[rN_1 \left(1 - \frac{N_1}{K}\right)(N_1 - m) - qN_1 \right].$$

By proceeding the same disretization process, for $t \in [nh, (n+1)h)$, $t/h \in [n, n+1)$, we have

$$N_{n+1} = N_n(nh) + \frac{(t-nh)^{\alpha}}{\Gamma(1+\alpha)} \left[rN_n(nh) \left(1 - \frac{N_n(nh)}{K}\right)(N_n(nh) - m) - qN_n(nh) \right]. \tag{8}$$

For $t \to (n+1)h$, eq. (8) is reduced to a discrete-time fractional order logistic model with the Allee effect and proportional harvesting

$$N_{n+1} = N_n + \frac{h^{\alpha}}{\Gamma(1+\alpha)} N_n \left[r \left(1 - \frac{N_n}{K}\right)(N_n - m) - q \right] = f(N). \tag{9}$$

We remark that if $\alpha \to 1$ then eq. (9) is exactly the same as the Euler discretization of model (5).

3 Fundamental Concepts

To analyze the dynamical behavior such as the existence of fixed point, the local stability, and the occurrence of saddle-node and period-doubling bifurcation of the discrete-time model (9), the following definition and theorems are needed.

Definition 1 [46] Consider the following map

$$x(n+1) = f(x(n)). \tag{10}$$

A point x^* is said a fixed point of the map (10) if $f(x^*) = x^*$. If $\left|f'(x^*)\right| \neq 1$ then x^* is called a hyperbolic fixed point, and if $\left|f'(x^*)\right| = 1$ then x^* is called a nonhyperbolic fixed point.

Theorem 1 *[46] Let x^* be a hyperbolic fixed point of the map (10) where f is continuously differentiable at x^*. The following statements then hold true:*

(i) If $|f'(x^*)| < 1$, then x^* is locally asymptotically stable.
(ii) If $|f'(x^*)| > 1$, then x^* is unstable.

Theorem 2 *[46] Let x^* is a nonhyperbolic fixed point of the map (10) satisfying $f'(x^*) = 1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at x^*, then the following statements hold:*

(i) If $f''(x^*) \neq 0$, then x^* unstable (semistable).
(ii) If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then x^* unstable.
(iii) If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* locally asymptotically stable.

Definition 2 [46] The Schwarzian derivative, Sf, of a function f is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2.$$

Particularly, if $f'(x^*) = -1$ then

$$Sf(x^*) = -f'''(x^*) - \frac{3}{2}\left[f''(x^*)\right]^2.$$

Theorem 3 *[46] Let x^* is a hyperbolic fixed point of the map (10) satisfying $f'(x^*) = -1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at x^*, then the following statements hold:*

(i) If $Sf(x^*) < 0$ then x^* is locally asymptotically stable.
(ii) If $Sf(x^*) > 0$ then x^* is unstable.

Theorem 4 *(The existence of Saddle-Node Bifurcation [46]) Suppose that $x_{n+1} = f(\mu, x_n)$ is a C^2 one-parameter family of one-dimensional maps, and x^* is a fixed point with $f'(\mu, x) = 1$. Assume further that*

$$\frac{\partial f}{\partial \mu}(\mu^*, x^*) \neq 0 \text{ and } \frac{\partial^2 f}{\partial x^2}(\mu^*, x^*) \neq 0.$$

Then there exists an interval I around x^* and a C^2 map $\mu = p(x)$, where $p : I \to \mathbb{R}$ such that $p(x^*) = \mu^*$, and $f(p(x), x) = x$. Moreover, if $\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}|_{(\mu^*, x^*)} < 0$, the fixed points exist for $\mu > \mu^*$ and if $\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}|_{(\mu^*, x^*)} > 0$, the fixed points exist for $\mu < \mu^*$.

Theorem 5 *(The existence of period-doubling bifurcation [46]) Suppose that $x_{n+1} = f(\mu, x_n)$ is a C^2 one-parameter family of one-dimensional maps, and x^* is a fixed point. Assume that*

(i) $\frac{\partial f}{\partial x}(\mu^*, x^*) = -1.$
(ii) $\frac{\partial^2 f}{\partial \mu \partial x}(\mu^*, x^*) \neq 0.$

Then there is an interval I about x^* and a function $p : I \to \mathbb{R}$ such that $f_{p(x)}(x) \neq x$ but $f^2_{p(x)} = x$.

4 Analytical Results

We explore some analytical results here such as the existence of fixed points, their local stability, and the existence of some bifurcations namely saddle-node and period-doubling bifurcations. Since the map (9) is constructed by PWCA with step-size (h), the analytical process is then investigated by considering the impact of h. Some analytical results also examine the influence of the harvesting (q) on the dynamics of the given map.

4.1 The Existence of Fixed Point

Based on Definition (1), the fixed point of the map (9) is obtained by solving the following equation

$$N = N + \frac{h^\alpha}{\Gamma(1+\alpha)} N \left[r\left(1 - \frac{N}{K}\right)(N - m) - q \right]. \tag{11}$$

The solutions of eq. (11) is described as follows:

(i) The extinction of population fixed point $N_0^* = 0$ which always exists.
(ii) The non-zero fixed points $N_{1,2}^*$ which are the positive solutions of the following quadratic polynomial

$$N^2 - (m+K)N + mK + \frac{qK}{r} = 0. \tag{12}$$

The solutions of eq. (12) are

$$\begin{aligned} N_1^* &= \frac{m+K}{2} + \frac{\sqrt{(q^*-q)rK}}{r}, \\ N_2^* &= \frac{m+K}{2} - \frac{\sqrt{(q^*-q)rK}}{r}, \end{aligned} \tag{13}$$

where $q^* = \frac{(m-K)^2 r}{4K} > 0$. The existence of non-zero fixed points (13) is shown by Theorem 6.

Theorem 6 *(i) If $q > q^*$ then the non-zero fixed point of the map (9) does not exist.*
(ii) If $q = q^*$ then there exists a unique non-zero fixed point $N^* = \frac{m+K}{2}$ of the map (9).

(iii) If $q < q^*$, then there exist two non-zero fixed points, namely $N^*_{1,2}$ of the map (9).

Proof (i) It is easy to confirm that if $q > q^*$ then the solutions of eq. (12) are a pair of complex conjugate numbers.

(ii) For $q = q^*$, we have $N^* = N^*_1 = N^*_2 = \frac{m+K}{2}$. Hence, N^* is the only positive fixed point of the map (9).

(iii) If $q < q^*$ then $N^*_{1,2} \in \mathbb{R}$. Because $N^*_1 N^*_2 = mK + \frac{qK}{r} > 0$ and $N^*_1 + N^*_2 = m + K > 0$, then N^*_1 and N^*_2 are obviously positive, showing that there are two non-zero fixed points.

□

4.2 Local Stability

Now, the dynamical behaviors of the map (9) around each fixed point are investigated. The following Theorems 7 to 10 are presenting to describe the local dynamics of Fixed points N^*_i, $i = 0, 1, 2$ and N^*. The complete dynamics including local stability, unstable condition, and nonhyperbolic properties for each fixed point are studied by employing Theorems 1 to 3. In this respect, all dynamical properties are expressed in step-size (h) and harvesting rate (q) to simplify the mathematical terms.

Theorem 7 *Let's denote $\hat{q} = \frac{(2m^2+3mK+2K^2)r}{K}$ and $h_0 = \sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{mr+q}}$. Then the following statements hold:*

(i) if $0 < h < h_0$ then N^*_0 is locally asymptotically stable,
(ii) if $h > h_0$ then N^*_0 is unstable, and
(iii) if $h = h_0$ then N^*_0 is nonhyperbolic fixed point. Furthermore if
(iii.a) $q < \hat{q}$ then N^*_0 is locally asymptotically stable, and
(iii.b) $q > \hat{q}$ then N^*_0 is unstable.

Proof By evaluating $f'(N)$ at N^*_0, we obtain

$$f'(N^*_0) = 1 - \frac{h^\alpha(mr+q)}{\Gamma(1+\alpha)} = 1 - 2\left(\frac{h}{h_0}\right)^\alpha.$$

(i) If $0 < h < h_0$ then $0 < (h/h_0)^\alpha < 1$, which implies $|f'(N^*_0)| < 1$. Based on Theorem 1, we have a locally asymptotically stable N^*_0.

(ii) If $h > h_0$ then $(h/h_0)^\alpha > 1$, so that $f'(N^*_0) < -1$. Theorem 1 states that N^*_0 is an unstable fixed point.

(iii) For $h = h_0$, we have $f'(N^*_0) = -1$, i.e., N^*_0 is nonhyperbolic fixed point. The Schwarzian derivative of map $f(N)$ at N^*_0 is

$$Sf(N^*_0) = \frac{h^\alpha}{\Gamma(1+\alpha)}\left[\frac{6r}{K}\right] - \frac{3}{2}\left[\frac{h^\alpha}{\Gamma(1+\alpha)}\left[\frac{2(m+K)r}{K}\right]\right]^2$$

$$
\begin{aligned}
&= \frac{6rh^\alpha}{K\Gamma(1+\alpha)}\left[1 - \frac{h^\alpha}{K\Gamma(1+\alpha)}(m+K)^2 r\right] \\
&= \frac{12r}{(mr+q)K}\left[1 - \frac{2(m+K)^2 r}{(mr+q)K}\right].
\end{aligned}
$$

If $q < \hat{q}$ then $Sf(N_0^*) < 0$, and thus N_0^* is locally asymptotically stable. On the contrary, if $q > \hat{q}$ then $Sf(N_0^*) > 0$, showing N_0^* is unstable. Thus, Theorem 7 is completely proven.

□

Theorem 8 *The non-zero fixed point N^* is semistable.*

Proof The derivative of $f(N)$ at N^* is

$$
\begin{aligned}
f'(N^*) &= 1 + \frac{h^\alpha}{\Gamma(1+\alpha)}\left[-\frac{3r}{K}\left(\frac{m+K}{2}\right)^2 + \frac{2r(m+K)}{K}\left(\frac{m+K}{2}\right) - mr - q\right] \\
&= 1 + \frac{h^\alpha}{\Gamma(1+\alpha)}\left[\frac{(m+K)^2 r - 4mrK}{4K} - q\right] \\
&= 1 + \frac{h^\alpha}{\Gamma(1+\alpha)}\left[q^* - q\right].
\end{aligned}
$$

N^* exists when $q = q^*$. Clearly that $f'(N^*) = 1$, and therefore N^* is a nonhyperbolic fixed point. By direct calculations, we can show that

$$
\begin{aligned}
f''(N^*) &= \frac{h^\alpha}{\Gamma(1+\alpha)}\left[-\frac{6rN}{K} + 2\left(1 + \frac{m}{K}\right) r\right] \\
&= \frac{h^\alpha}{\Gamma(1+\alpha)}\left[-\frac{6r}{K}\frac{m+K}{2} + \frac{2(m+K)r}{K}\right] \\
&= \frac{h^\alpha}{\Gamma(1+\alpha)}\left[-\frac{3(m+K)r}{K} + \frac{2(m+K)r}{K}\right] \\
&= -\frac{(m+K)rh^\alpha}{K\Gamma(1+\alpha)} < 0.
\end{aligned}
$$

Since $f''(N^*) \neq 0$, Theorem 2 says that the fixed point N^* is semistable. □

Theorem 9 *Suppose that:*

$$
\begin{aligned}
h_1 &= \sqrt[\alpha]{\frac{2K\Gamma(1+\alpha)}{2(q^* - q)K + (m+K)\sqrt{(q^* - q)rK}}}, \\
\hat{h} &= \frac{(m+K)r + 6\sqrt{(q^* - q)K}}{\sqrt{4(q^* - q)rK + 2r(m+K)\sqrt{(q^* - q)rK}}}.
\end{aligned}
$$

The local stability of N_1^* is described as follows.

(i) If $0 < h < h_1$ then N_1^* is locally asymptotically stable.
(ii) If $h > h_1$ then N_1^* is unstable.

(iii) If $h = h_1$ and
(iii.a) If $\hat{h} > 1$ then N_1^* is locally asymptotically stable.
(iii.b) If $\hat{h} < 1$ then N_1^* is unstable.

Proof It is obvious to show that

$$\begin{aligned} f'(N_1^*) &= 1 + \tfrac{h^\alpha}{\Gamma(1+\alpha)} \Big[- \Big(\tfrac{3}{4K}(m+K)^2 r + 3(q^* - q) + \tfrac{3}{K}(m+K)\sqrt{(q^*-q)rK} \Big) \\ &\quad + \tfrac{1}{K}(m+K)^2 r + \tfrac{2}{K}(m+K)\sqrt{(q^*-q)rK} - (mr+q) \Big] \\ &= 1 - \tfrac{h^\alpha}{K\Gamma(1+\alpha)} \Big[2(q^*-q)K + (m+K)\sqrt{(q^*-q)rK} \Big] \\ &= 1 - \tfrac{h^\alpha}{K\Gamma(1+\alpha)} \Big[\tfrac{2K\Gamma(1+\alpha)}{h_1^\alpha} \Big] \\ &= 1 - 2\left(\tfrac{h}{h_1}\right)^\alpha . \end{aligned}$$

Hence, we have the following observations:

(i) For $0 < h < h_1$, we have $|f'(N_1^*)| < 1$. According to Theorem 1, the non-zero fixed point N_1^* is locally asymptotically stable.

(ii) If $h > h_1$ then we get $f'(N_1^*) < -1$. Thus, N_1^* is an unstable fixed point (see Theorem 1).

(iii) Clearly that $f'(N_1^*) = -1$ whenever $h = h_1$, which shows that N_1^* is non-hyperbolic fixed point. The Schwarzian derivative of $f(N)$ at N_1^* is given by

$$\begin{aligned} Sf(N_1^*) &= \tfrac{h^\alpha}{\Gamma(1+\alpha)} \Big[\tfrac{6r^2}{K} \Big] - \tfrac{3}{2} \Big[- \tfrac{h^\alpha}{\Gamma(1+\alpha)} \Big[\tfrac{(m+K)r}{K} + \tfrac{6}{K}\sqrt{(q^*-q)K} \Big] \Big]^2 \\ &= \tfrac{rh^\alpha}{K\Gamma(1+\alpha)} \Big[6r - \tfrac{3}{2K} \Big[\tfrac{h^\alpha}{\Gamma(1+\alpha)} \Big] \Big[(m+K)r + 6\sqrt{(q^*-q)K} \Big]^2 \Big] . \end{aligned}$$

We can easily check that if $\hat{h} > 1$ then $Sf(N_1^*) < 1$ and if $\hat{h} < 1$ then $Sf(N_1^*) > 1$. Therefore, the stability of the nonhyperbolic fixed point is explained. Finally, all of the stability conditions of fixed point N_1^* are completely determined.

□

Theorem 10 *The non-zero fixed point $N_2^* = \frac{m+K}{2} - \frac{\sqrt{(q^*-q)rK}}{r}$ is always unstable.*

Proof To investigate the stability of N_2^*, we evaluate $f'(N)$ at N_2^*:

$$f'(N_2^*) = 1 + \frac{(q^*-q)h^\alpha}{\Gamma(1+\alpha)} \left[\frac{(m+K)\sqrt{r}}{\sqrt{(q^*-q)K}} - 2 \right].$$

By simple algebraic manipulations, we can show that $\frac{(m+K)\sqrt{r}}{\sqrt{(q^*-q)K}} > 2$. Thus, $f'(N_2^*)$ is always a positive constant, which means N_2^* is always an unstable fixed point. □

4.3 Bifurcation Analysis

From the previous analysis, we have a non-hyperbolic fixed point N^* when $q = q^*$, indicating the possibility of the occurrence of saddle-node bifurcation. Moreover, the occurrence of period-doubling bifurcation is also indicated around the non-hyperbolic fixed point N_1^* when $h = h_1$. Thus, in this section, we study the existence of saddle-node and period-doubling bifurcations. The saddle-node bifurcation is a phenomenon that two fixed points with opposite signs of stability merge into a unique semi-stable fixed point and finally disappear when a parameter is varied, while the period-doubling bifurcation is a phenomenon that a single fixed point losses its stability accompanied by the emergence of a period-2 solution when a parameter is varied [46]. As results, we have Theorems 11 and 12.

Theorem 11 *The non-zero fixed point N^* undergoes a saddle-node bifurcation when q crosses the critical values $q^* = \frac{(m-K)^2 r}{4K}$.*

Proof It was shown previously that N^* does not exist if $q > q^*$. When $q = q^*$ we have a semistable fixed point N^*; and if $q < q^*$, then there exists two non-zero fixed points. By straightforward calculations, we have $\frac{\partial f(N^*)}{\partial N} = 1$, $\frac{\partial f(N^*)}{\partial q} = -\frac{h^\alpha}{\Gamma(1+\alpha)} \frac{m+K}{2} < 0$, and $\frac{\partial^2 f(N^*)}{\partial N^2} = -\frac{(m+K)rh^\alpha}{K\Gamma(1+\alpha)} < 0$. Thus, according to Theorem 4, the fixed point N^* undergoes a saddle-node bifurcation when q crosses the critical values $q^* = \frac{(m-K)^2 r}{4K}$. Moreover, the fixed points exist when $q \le q^*$ because $\frac{\partial f(N^*)}{\partial q} \frac{\partial^2 f(N^*)}{\partial N^2} > 0$. □

Theorem 12 *The non-zero fixed point N_1^* undergoes a period-doubling bifurcation when h crosses the critical value $h_1 = \sqrt[\alpha]{\frac{2K\Gamma(1+\alpha)}{2(q^*-q)K+(m+K)\sqrt{(q^*-q)rK}}}$.*

Proof From the proof of Theorem 9 we have that if $h = h_1$ then $\frac{\partial f(N_1^*)}{\partial N} = -1$. By performing some algebraic calculations, we also have

$$\begin{aligned}\frac{\partial^2 f(N_1^*)}{\partial h \partial N} =& \frac{\alpha h^{\alpha-1}}{\Gamma(1+\alpha)} \left[r \left(\frac{m+K}{2} + \frac{\sqrt{(q^*-q)rK}}{r} \right) \right. \\ & \left. \left(1 - \frac{2}{K} \left(\frac{m+K}{2} + \frac{\sqrt{(q^*-q)rK}}{r} \right) + \frac{m}{K} \right) \right] \\ =& -\frac{r\alpha h^{\alpha-1}}{\Gamma(1+\alpha)} \left(\frac{m+K}{2} + \frac{\sqrt{(q^*-q)rK}}{r} \right) \left(\frac{2}{K} \frac{\sqrt{(q^*-q)rK}}{r} \right).\end{aligned} \tag{14}$$

N_1^* exists if $q \neq q^*$ and thus we have $\frac{\partial^2 f(N_1^*)}{\partial h \partial N} \neq 0$. According to Theorem 5, there appears a solution of period-2 when h passes through h_1. Hence, the occurrence of period-doubling bifurcation in the map (9) is completely proven. □

Theorem 12 states that the period-doubling bifurcation in the map (9) can be achieved by varying the step-size h. However such bifurcation can also be realized by setting a fixed value of h and other parameters while varying a certain parameter. In the following Section, we give an example of period-doubling bifurcation which is driven by the constant of harvesting (q).

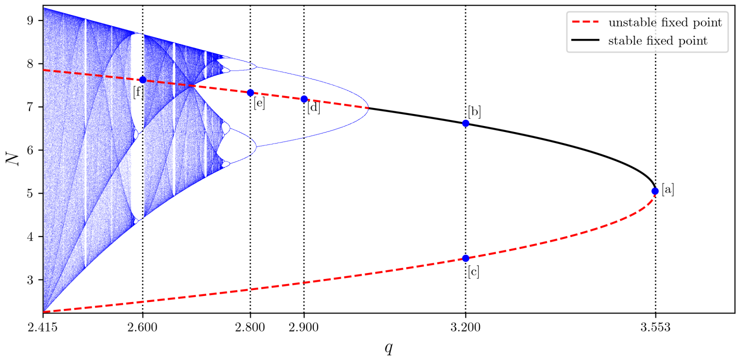
5 Numerical Results

In this section, we present some numerical simulations of the map (9) not only to support the previous analytical findings but also to show more dynamical behaviors of the map (9). Numerical simulations are given by considering some biological and mathematical aspects such as the influence of the harvesting, the step-size (h), the Allee effect (m), and the order$-\alpha$. To support the numerical simulations, a desktop PC is used based on AMD Ryzen 5 3400G 3.7GHz, 16 GB RAM, and AMD Radeon RX580 8GB DDR5 VGA card. We also use an open source software called Python 3.9 to generate all of the given figures. Due to the field data limitation, we use hypothetical parameter values for the numerical simulations. General parameter values are given as follows.

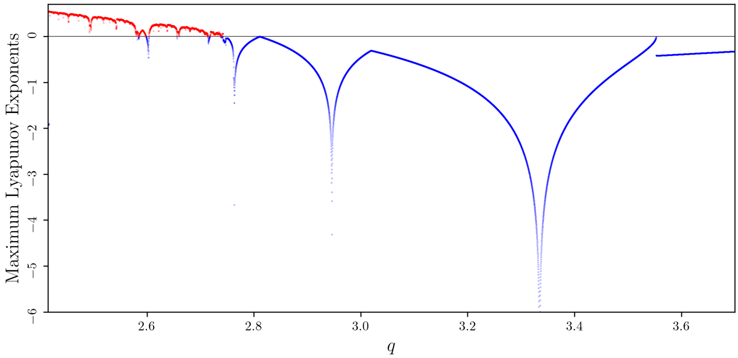
$$r = 1.45,\ K = 10,\ m = 0.1,\ q = 0.32,\ \alpha = 0.8,\ \text{and } h = 0.4. \tag{15}$$

5.1 The influence of the Harvesting Rate

The numerical simulations in this subsection are using parameter set (15) and vary the value of the harvesting rate (q). According to Theorem 6, map (9) with parameter set (15) has critical value $q^* \approx 3.5527$ such that map (9) does not have a non-zero fixed point if $q > q^*$. When $q = q^*$, map (9) has a unique non-zero fixed point $N^* = 5.05$ which is a semistable fixed point, see Theorem 8. Furthermore, if $q < q^*$, then there are two non-zero fixed points, namely N_1^* and N_2^*. By taking $h = 0.4$ and using Theorem 9, we can show that N_1^* is asymptotically stable if $q_1 = 3.0191 \lesssim q < q^*$. On the other hand, Theorem 10 states that N_2^* is always unstable. Since we take $h = 0.4$, Theorem 12 states that the fixed point N_1^* undergoes a period-doubling bifurcation when q crosses q_1 from the right. To see these dynamical behaviors, we plot in fig. 1a the bifurcation diagram of the map (9) with parameter set (15) and $h = 0.4$ for $2.415 \leq q \leq 3.7$. Clearly that this bifurcation diagram fits perfectly with the results of our previous analysis. Indeed, fig. 1a shows that N^* (labeled as [a]) is semistable, see also the Cobweb diagram shown in fig. 2a. As the value of q decreases from q^*, the non-zero fixed point is split into two non-zero fixed points where one of them is stable in the specified interval of q, while the other fixed point is unstable. Such stability properties can also be seen in the Cobweb diagrams in figs. 2b and 2c, which corresponds to points [b] and [c] in fig. 1a, respectively. We also observe numerically the appearance of a period-doubling route to chaos (flip bifurcation) as q decreases. If we further



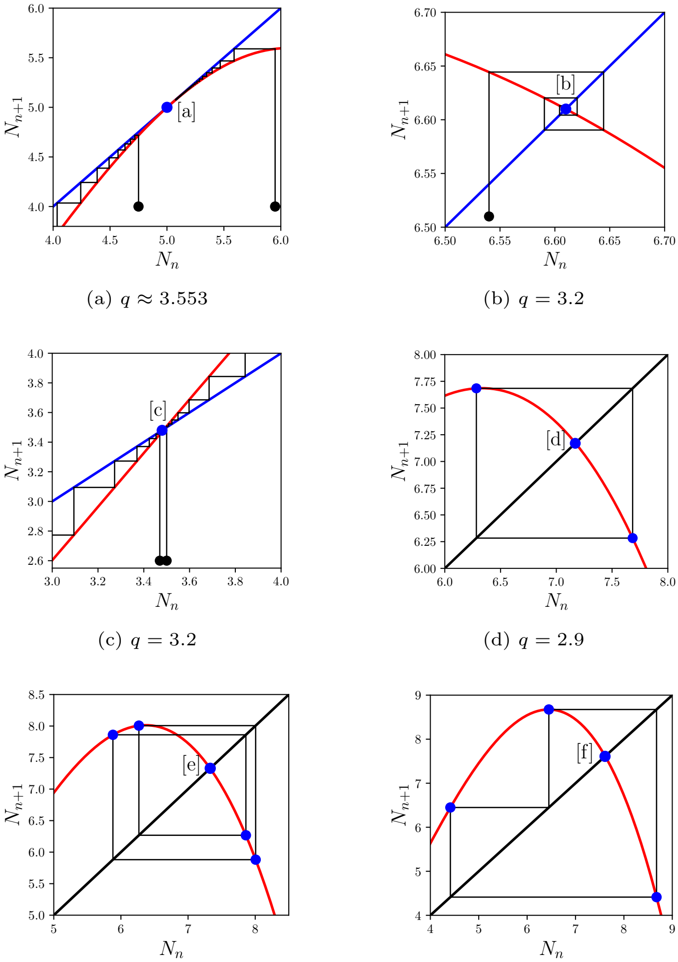
(a) Bifurcation diagram



(b) Maximum Lyapunov exponents

Fig. 1: Bifurcation diagram and its corresponding maximum Lyapunov exponents of the map (9) with parameter set (15) and $2.415 \leq q \leq 3.7$

decrease the value of q, then there appears a stable solution of period-2 when q passes through q_1. The appearance of a stable period-doubling solution, as well as a solution of period-3, are seen in fig. 1a (see e.g. point [d], [e], and [f], respectively, and their corresponding diagram Cobweb in figs. 2d to 2f). The appearance of the period-3 solution indicates that our system exhibits chaotic dynamics [47]. The existence of chaotic dynamics can also be determined from the Lyapunov exponent. A system exhibits chaotic dynamics if it has positive maximum Lyapunov exponents. The maximum Lyapunov exponents which correspond to fig. 1a is depicted in fig. 1b. It is clearly seen that our system has positive maximum Lyapunov exponents, showing the existence of chaotic dynamics in the map (9) which is controlled by the constant of harvesting (q).



(a) $q \approx 3.553$ (b) $q = 3.2$

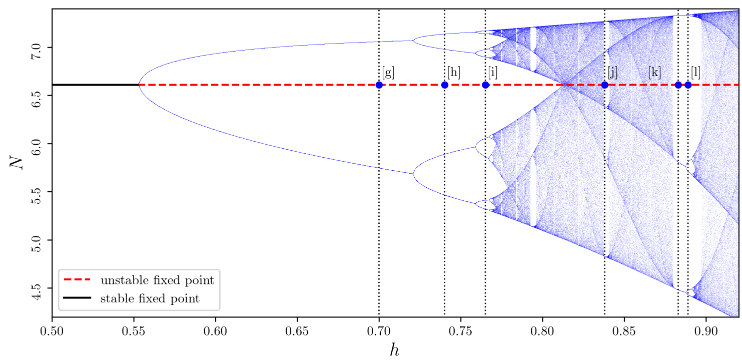
(c) $q = 3.2$ (d) $q = 2.9$

(e) $q = 2.8$ (f) $q = 2.6$

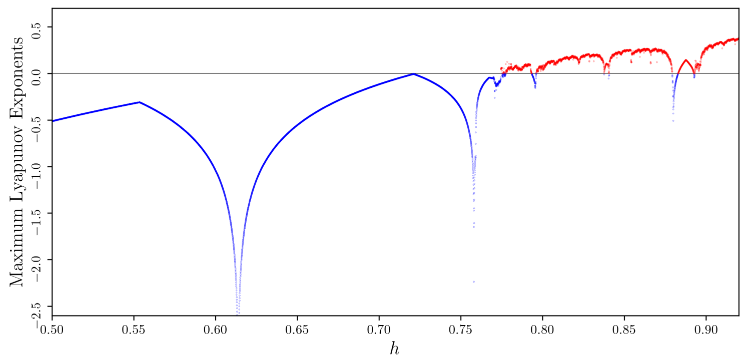
Fig. 2: Cobweb diagrams of the map (9) with parameter set (15)

5.2 The influence of the Step-Size

To describe the existence of period-doubling bifurcation driven by the step-size h numerically, we perform simulations using the parameter set (15) and $0.5 \leq h \leq 0.985$. Map (9) with these parameter values has two non-zero fixed points, namely $N_1^* \approx 6.61$ and $N_2^* \approx 3.49$. N_2^* is unstable while N_1^* is stable if $0 < h < h_1 \approx 0.553$. N_1^* losses its stability via period-doubling bifurcation when h crosses h_1. These dynamics are seen in the bifurcation diagram, see fig. 3a. Increasing the value of h may destroy the stability of N_1^* and the system is convergent to a stable period-2 solution. Further increasing the value of h leads to a stable period-4 cycle, and so on. To give a more detailed view, we



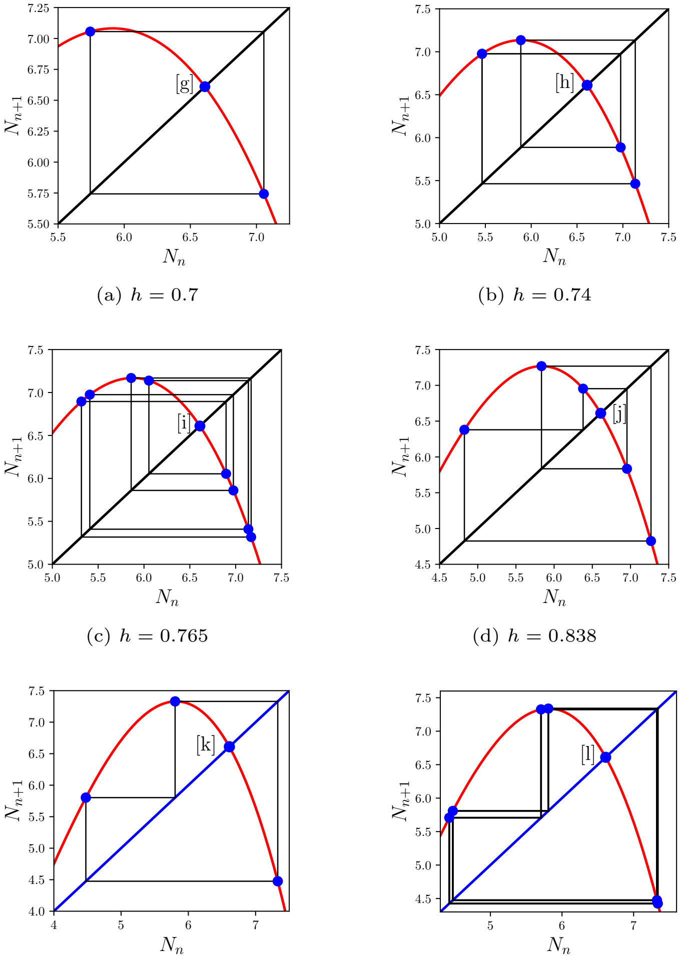
(a) Bifurcation diagram



(b) Maximum Lyapunov exponents

Fig. 3: Bifurcation diagram and its corresponding maximum Lyapunov exponents of the map (9) with parameter set (15) and $0 \leq h \leq 0.92$

plot Cobweb diagrams in fig. 4 which correspond to some solutions around the fixed points labeled as [g - l] in fig. 3a. When $h = 0.7$, we have a stable period-2 cycle near the non-zero fixed point [g], see fig. 4a. Each of the two solutions splits into two solutions respectively and becomes a stable period-4 solution around fixed point [h] when $h = 0.74$ (fig. 4b); and consecutively for $h = 0.765$ we have a stable period-8 cycle near fixed point [i], see fig. 4c. Moreover, at $h = 0.838, 0.883, 0.889$ we have respectively a stable period-5 cycle around fixed point [j], a stable period-3 cycle around fixed point [k], and a stable-period-6 cycle around fixed point [l], see their Cobweb diagrams in figs. 4d to 4f. Hence, the step-size h is an important parameter that significantly affects the dynamics of the map (9). In this case, the map (9) exhibits a period-doubling bifurcation route to chaos driven by parameter h. Furthermore, the appearance of positive maximum Lyapunov exponents depicted in fig. 3b which corresponds to the bifurcation diagram in fig. 3a clearly shows the existence of chaotic behavior in the system.



(a) $h = 0.7$ (b) $h = 0.74$

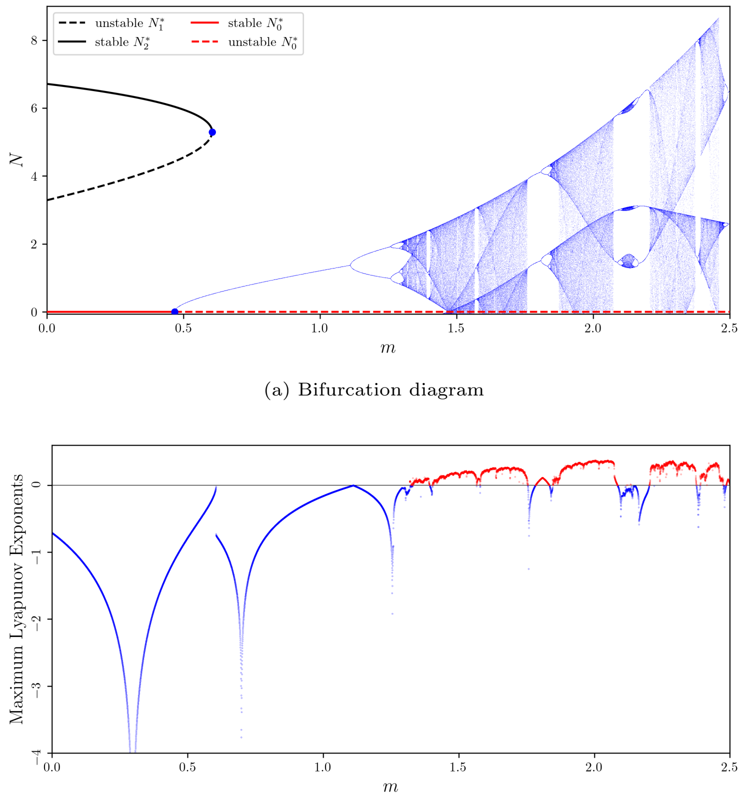
(c) $h = 0.765$ (d) $h = 0.838$

(e) $h = 0.883$ (f) $h = 0.889$

Fig. 4: Cobweb diagrams of the map (9) with parameter set (15)

5.3 The Influence of the Allee Effect

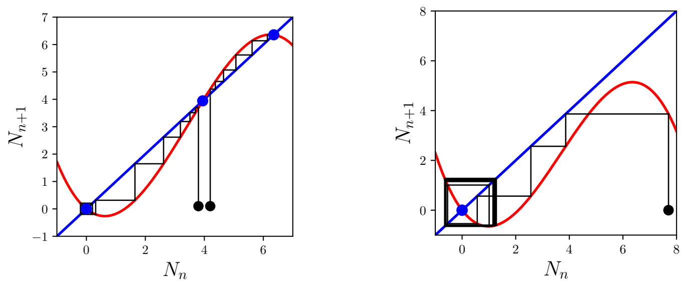
To show the influence of the Allee effect, we use the parameter set (15) and vary the values of m in the interval $0 \leq m \leq 2.5$. From Equation (13), we compute numerically that N_1^* and N_2^* exist for interval $0 \leq m \lessapprox 0.6045$. Based on Theorems 8 to 10, the stability of N_1^* and N_2^* has the different sign for $0 \leq m < 0.6045$ and finally merge into a semi-stable fixed point $N^* \approx 5.29671$ when $m \approx 0.6045$. When m crosses 0.6045, N^* disappears and N_0^* becomes the only fixed point of the map (9). These phenomena indicate the occurrence of saddle-node bifurcation driven by the Allee effect (m). According to Theorem 7, we also have that N_0^* is locally asymptotically stable for $m < 0.467$



(a) Bifurcation diagram

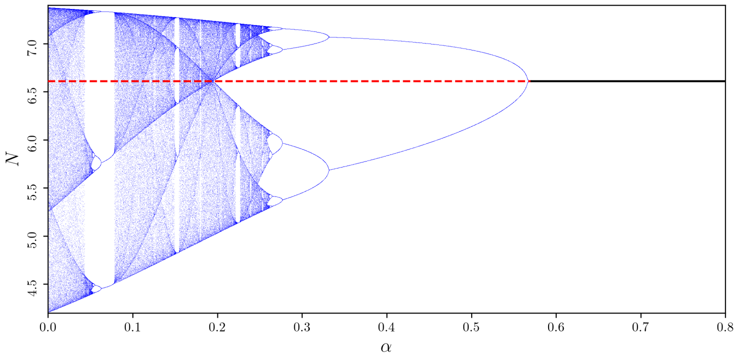
(b) Maximum Lyapunov exponents

Fig. 5: Bifurcation diagram and its corresponding maximum Lyapunov exponents of the map (9) with parameter set (15) and $0 \leq m \leq 2.5$

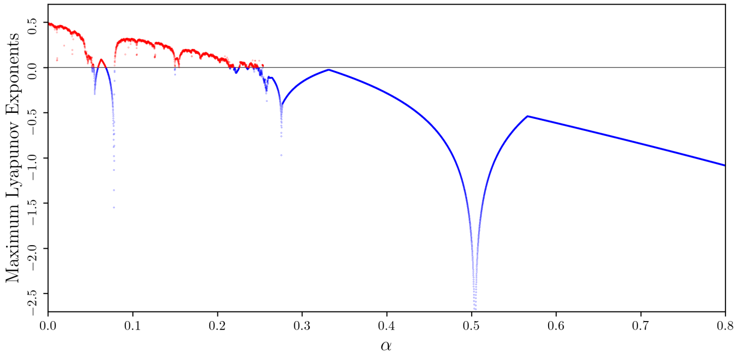


(a) $m = 0.3$ (b) $m = 1$

Fig. 6: Cobweb diagrams of the map (9) with parameter set (15)



(a) Bifurcation diagram



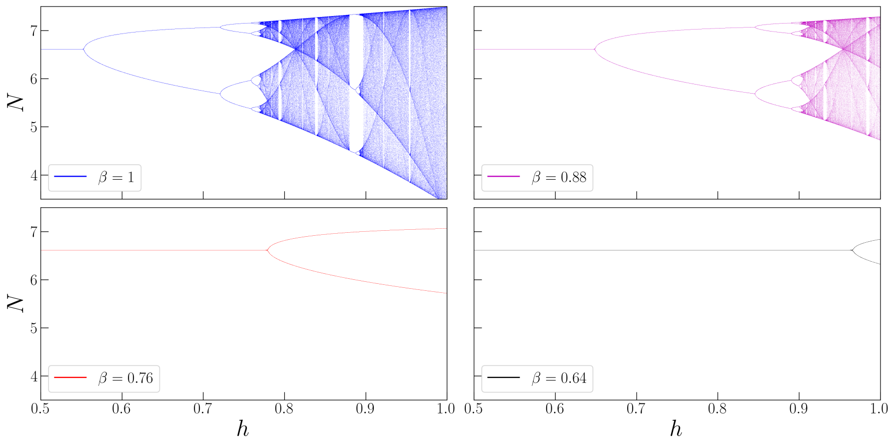
(b) Maximum Lyapunov exponents

Fig. 7: Bifurcation diagram and its corresponding maximum Lyapunov exponents of the map (9) with parameter set (15) and $0 \leq \alpha \leq 0.8$

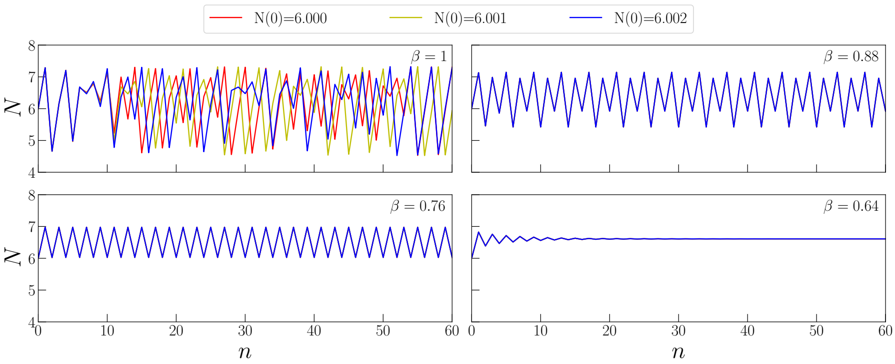
and losses its stability via period-doubling bifurcation when m crosses 0.467. These complex dynamics are shown in fig. 5a and its corresponding maximum Lyapunov exponents are depicted in fig. 5b which confirms the existence of chaotic behavior on the map (9). One interesting condition is also shown for some values of m. For $0 < m < 0.467$, the map (9) passes through a bistability condition. N_0^* and N_1^* are locally asymptotically stable simultaneously and hence the solution of the map is sensitive to the initial value. See the Cobweb diagrams in fig. 6. When $m = 0.3$, two nearby initial values are convergent to different fixed points. When the Allee effect increases to $m = 1$, the solution converges to a period-2 solution around N_0^*.

5.4 The Influence of the order-α

As the impact of the discretization process, we have a parameter α on map (9) which is derived from the order of the derivative of the continuous model



(a) Bifurcation diagram



(b) Solutions for $h = 0.877$

Fig. 8: Bifurcation diagrams of controlled map (17)

as the memory effect. Again, we use the parameter set (15) and varying α. As result, we have a bifurcation diagram and maximum Lyapunov exponents depicted in Figure 7. The given dynamics are quite similar to the impact of the step-size but in different directions. If increasing h may change the dynamics of N_1^* from locally asymptotically stable to periodic solution via period-doubling bifurcation, different dynamics direction presented by α where if its value increase, the unstable N_1^* becomes locally asymptotically stable via period-doubling bifurcation. Some chaotic behavior indicated by positive Lyapunov exponents disappears becomes periodic orbits and is finally convergent to N^* when α crosses 0.5708.

6 Hybrid Control Strategy

In this section, a method namely the hybrid control strategy is presented. This method is a combination of state feedback and parameter perturbation which is used for controlling bifurcation in a discrete system [48–51]. We first define a map (9) as follows.

$$N_{n+1} = f(N_n, \zeta), \tag{16}$$

where $N \in \mathbb{R}$ is the population density and, $F(N_n, \zeta)$ is the right hand side of map (9) with bifurcation parameter $\zeta \in \mathbb{R}$. It can be revisited from analytical and numerical results that when h and q are varies in some range, the map (9) passes through a series of period-doubling bifurcations where the route to chaos. By obeying state feedback and parameter perturbation to the map (9), we obtain the control map as follows.

$$N_{n+1} = \beta f(N_n, \zeta) + (1 - \beta)N_n = F(N, \beta), \tag{17}$$

where $\beta \in [0, 1]$ denotes the external control parameter for map (17). We can easily show that the map (9) and (17) have similar fixed points. From Theorem 12, N_1^* is the fixed point which undergoes a period-doubling bifurcation. Particularly, From Theorem 1 in [51], the m-periodic orbit of control map (17) is also similar to the original map (9). Now, we will show that by setting β and varying h, the occurrence of period-doubling bifurcation can be delayed or even eliminated. From the control map (17) we have $F'(N_1^*) = 1-2\beta\left(\frac{h}{h_1}\right)^{\alpha}$ and $\frac{\partial^2 F(N_1^*)}{\partial h \partial N} = \frac{\partial^2 f(N_1^*)}{\partial h \partial N} < 0$. According to Theorem 5, the control map (17) also undergoes period-doubling bifurcation for the similar fixed point with map (9). The difference lies in the bifurcation point where the map (9) is $h = h_1$ while the control map (17) is $h = \frac{h_1}{\sqrt[\alpha]{\beta}}$. This means if β decreases then the bifurcation point increase which means the series of periodic solutions are delayed. For example, by setting the parameter values as in eq. (15) and $\beta = 0.64, 0.76, 0.88, 1$, the occurrence of bifurcation is delayed and period-3 solutions disappear. See Figure 8a. We also check the chaotic solution near the period-3 solution. For $h = 0.887$, three quite close initial conditions $N(0) = 6, 6.001, 6.002$ is given and portray the solutions in Figure 8b. The chaotic interval which occurs for $\beta = 1$ becomes a periodic solution for $\beta = 0.76, 0.88$, and finally, converges to N_1^* when $\beta = 0.64$.

7 Conclusion

A discrete-time fractional-order logistic model with the Allee effect and proportional harvesting has been constructed and investigated dynamically. The discrete-time model is derived by applying the PWCA method to the Caputo fractional order modified logistic model. The local stability for each fixed point is successfully investigated completely for hyperbolic and nonhyperbolic fixed points by obeying the stability theorem along with the Schwarzian derivative. Furthermore, it was shown analytically that the obtained discrete-time model exhibits a saddle-node bifurcation as well as period-doubling bifurcation. The key parameter in such bifurcations is the constant of harvesting (q) or the step-size (h). Numerical simulations with varying parameters q and h confirm our analytical results. The dynamics of the map are also studied numerically by varying the Allee threshold (m) and the order$-\alpha$ which also give the saddle-node and period-doubling bifurcations. Furthermore, the presented numerical results also showed the existence of period-doubling route chaos which is indicated by the positive Lyapunov exponents and the appearance of period-3 window. We then construct the control based on the hybrid control strategy method. It is shown that the occurrence of period-doubling can be delayed. The occurrence of the chaotic solution is also successfully eliminated when the control parameter is decreased.

Declarations

Author Contributions. All authors contributed equally to this article. They read and approved the final manuscript.

Data availability. Not applicable.

Conflict of interest. All authors declare that they have no conflicts of interest.

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Author’s Response to Reviewers‘ Comments

Title : Bifurcation and Chaos in a Discrete-Time Fractional-Order Logistic Model with Allee Effect and Proportional Harvesting
Manuscript ID : IJDY-D-22-00714

Dear Prof. Jian-Qiao Sun,

Editor in Chief
Of International Journal of Dynamics and Control

We have revised our manuscript based on the suggestion and advice from both two reviewers. To facilitate and to make it easier to find the parts we have revised, we have colored in blue the modified parts and provided the details below.

Reviewer 1

This paper considers a Caputo fractional-order logistic model with the Allee effect and proportional harvesting. They implement the piecewise constant arguments method to discretize the fractional model. Theoretically, the existence and stability conditions of the fixed points of the discrete model are given. Furthermore, the existence of saddle-node bifurcation and period-doubling bifurcation is proven. Finally, these analytical results are then confirmed by some numerical simulations via bifurcation diagrams, Cobweb diagrams, and maximal Lyapunov exponents. Some questions and comments need to be considered by the authors:

1. This paper studies a fractional-order model. Has the author considered the comparison between this model and the integer-order model?

> In this paper, we only study the discrete-time model constructed from its fractional-order model. We didn’t compare the model with those two operators by considering the fractional-order model in a one-dimensional model has poor dynamics since $|\arg(\lambda)|$ of the equilibrium point only has three conditions i.e., positive, negative, and zero. Those dynamics are qualitatively the same as the first-order model, and the difference only lies in the convergence rate. In another hand, the discrete-time model proposed in this paper has rich dynamics such as the existence of period-doubling bifurcation and the occurrence of chaotic dynamics. We add this explanation on page 3 lines 47-58.

2. The factors considered in the model established in this paper are fractional order, the Allee effect, and Harvesting. However, the author did not reflect on the impact of fractional order and the Allee effect on the model dynamics in the numerical simulation. As far as I know, these two factors will also bring about complex dynamic phenomena.

> We have added the influence of the Allee effect and the order$-\alpha$ to our manuscript. See pages 15-18 lines 269-300.

3. There are some minor mistakes in the manuscript that need the author's attention, for example, "conrolled" in the abstract, model (5) should not use commas, and the title and Y-axis of Fig.1 should be unified, etc.

> We have modified and revised some typos. For the figures, we use the sub-caption rather than title and hence the position for each sub-caption adapted to LaTeX templates.

Reviewer 2

Herein, the author(s) consider a Caputo fractional order logistic model with Allee effect and proportional harvesting. Especially, they implement the piecewise constant arguments method to discretize the fractional model. The dynamics of the obtained discrete-time model is then analyzed. Fixed points and their stability conditions are established. Also they show the existence of saddle-node and period doubling bifurcations in the discrete-time model. These analytical results are then confirmed by some numerical simulations via bifurcation diagram, Cobweb diagram and maximal Lyapunov exponents. The article needs a careful revision before the possibility of its consideration for publication in IJDY, the following comments should be addressed:

1. Extend the introductory section to cover the pros/cons of the work.

We extend the introduction on page 3 lines 47-58 to show some explanation about the based model such as first-order, fractional-order, and discrete-time models, and the reason prefers to study the discrete-time model.

2. A table of notation is required before Section 2.

We have added table 1 on page 2.

3. The target of each theorem should be reported before the theorem.

We have added some explanations for local dynamics for every fixed point on page 7 lines 135-141. We also explore more the description of bifurcation analysis on page 10 lines 179-189.

4. Mention the specs of the machine used to depict all figures as well as the platform used to generate these figures.

We have mentioned the specification of the PC along with the software for numerical simulations on page 11 lines 208-209.

5. Extend the conclusion section to cover the future extensions in this direction.

We have extended the conclusion based on the reviewers' suggestions. See the blue texts on pages 19-20 lines 322-340/

6. Update the review literature with recent relevant work in the field of numerical methods for handling Time-fractional problems, for instance, https://doi.org/10.1007/s40314-022-02096-7 and https://doi.org/10.3934/math.2022830

We have updated the references to cover relevant works to this model.

All of the reviewers' suggestion and advice have been revised and modified. We hope the revise version of this manuscript meets the requirement of the **International Journal of Dynamics and Control**.

Your Sincerely,
Authors

Hasan S. Panigoro <hspanigoro@ung.ac.id>

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Dear Mr. Panigoro,

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With best regards,

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Bifurcation and chaos in a discrete-time fractional-order logistic model with Allee effect and proportional harvesting

Hasan S. Panigoro[1] · Maya Rayungsari[2,3] · Agus Suryanto[2]

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Abstract

The Allee effect and harvesting always get a pivotal role in studying the preservation of a population. In this context, we consider a Caputo fractional-order logistic model with the Allee effect and proportional harvesting. In particular, we implement the piecewise constant arguments (PWCA) method to discretize the fractional model. The dynamics of the obtained discrete-time model are then analyzed. Fixed points and their stability conditions are established. We also show the existence of saddle-node and period-doubling bifurcations in the discrete-time model. These analytical results are then confirmed by some numerical simulations via bifurcation, Cobweb, and maximal Lyapunov exponent diagrams. The occurrence of period-doubling bifurcation route to chaos is also observed numerically. Finally, the occurrence of period-doubling bifurcation is successfully controlled using a hybrid control strategy.

Keywords Discrete-time fractional-order · Logistic map · Allee effect · Harvesting · Bifurcation · Chaos

Mathematics Subject Classification 34A08 · 39A28 · 39A30 · 92D40

1 Introduction

For the last decades, the discrete-time model gets a lot of great attentiveness from researchers in mathematical modeling, not only because of its capability in describing several phenomena such as physics, biomedicine, engineering, chemistry, and population dynamics but also due to the richness of the given dynamical patterns as well as the occurrence of bifurcations and chaotic solutions which very difficult to find in their continuous counterpart [1–6]. Particularly, the discrete-time model is successfully applied in population dynamics, especially in a single logistic growth modeling [7–10], the epidemic modeling [11–13], and the predator–prey interaction modeling [14–18]. Most of the models are discretized using Euler scheme [19–21] and nonstandard finite difference (NSFD) [22–24] which is popular for the discretization of the model with first-order derivative as the operator. Furthermore, for the model with fractional-order derivative, we have some numerical schemes to approximate the exact solution such in [25–28]. We also have the popular discretization process is given by piecewise constant arguments (PWCA) which were proposed by El-Sayed et al. [29] and applied by other researchers in different biological phenomena [30–34].

In this paper, we study and justify the dynamics of a discrete-time model constructed using PWCA from a fractional-order logistic growth model involving the Allee effect and harvesting. The model is given by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1-\frac{N}{K}\right)(N-m) - qN, \tag{1}$$

where $N(t)$ represents the population density at time t and all parameters are positive numbers with biological interpretations and are given in Table 1.

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Table 1 Biological interpretation for each Parameter

Parameters	Biological interpretation
r	The intrinsic growth rate
K	The environmental carrying capacity
m	The Allee effect threshold
q	The harvesting rate

Notice that the Allee effect reduces the population growth rate when the population density is low (i.e., when $N < m$) as a result of several natural mechanisms such as intraspecific competition, cooperative anti-predator behavior, cooperative breeding, and limitation in finding mates. The positive growth rate occurs if the population density is in the interval $m < N < K$. For further explanation about the Allee effect, see [35–44].

To obtain the fractional-order model, we follow a similar way as in [14]. The first-order derivative at the left-hand side of model (1) is replaced with the fractional-order derivative ${}^{C}\mathcal{D}_t^{\alpha}$ which denotes the Caputo fractional derivative operator of order α defined by

$$ {}^{C}\mathcal{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} ds, \tag{2}$$

where α is the order of fractional derivative with $\alpha \in (0, 1]$ and $\Gamma(\cdot)$ is the Gamma function. Furthermore, by replacing the operator with equating the dimensions of time at the right-hand side, the following model is acquired.

$$ {}^{C}\mathcal{D}_t^{\alpha} N = r^{\alpha} N \left(1 - \frac{N}{K}\right)(N - m) - q^{\alpha} N. \tag{3}$$

Model (3) can be written as

$$ {}^{C}\mathcal{D}_t^{\alpha} N = \bar{r} N \left(1 - \frac{N}{K}\right)(N - m) - \bar{q} N, \tag{4}$$

where $\bar{r} = r^{\alpha}$ and $\bar{q} = q^{\alpha}$. Finally, by dropping $(\bar{\cdot})$, the fractional-order model for (1) is successfully obtained as follows.

$$ {}^{C}\mathcal{D}_t^{\alpha} N = r N \left(1 - \frac{N}{K}\right)(N - m) - q N. \tag{5}$$

As far as we are aware, both the fractional-order model and the discrete-time version of Eq. 5 have not been introduced and studied. Especially for fractional-order model (5), since the stability properties of equilibrium point refers to Matignon condition [45], the dynamics of the one-dimensional first-order and the fractional-order models are qualitatively the same because the $|\arg(\lambda)|$ of equilibrium point always in the real line. On the other hand, although in the one-dimensional model, the discrete-time model has more possible complex phenomena such as period-doubling bifurcation and chaotic behaviors which do not exist in its continuous ones. This means the one-dimensional continuous model has poor dynamics than the discrete-time model. Hence, for this case, studying the discrete-time model is more interesting and attractive. In this paper, we construct a discrete-time model by implementing the PWCA method for the model (5), and the dynamics of the obtained discrete-time model are then investigated. The layout of this paper is as follows. In Sect. 2, the model formulation is given by applying the PWCA method to get a discrete-time model. To support the analytical process, we provide some basic theoretical results in Sect. 3. In Sect. 4, some analytical results are provided such as the existence of fixed points, their local stability, and saddle-node and period-doubling bifurcations. In Sect. 5, we present some numerical simulations and show some interesting phenomena, such as bifurcation, Lyapunov exponent, and Cobweb diagrams which correspond to the previous theoretical results. We also present numerically a period-doubling route to chaos. A hybrid control strategy is applied to delay and eliminate the occurrence of period-doubling bifurcation and chaotic solution in Sect. 6. The conclusion of this work is given in Sect. 7.

2 Model formulation

By applying a similar procedure as in [29,30], we discretize model (5) with the PWCA method as follows

$$ \begin{aligned} {}^{C}\mathcal{D}_t^{\alpha} N(t) &= r N([t/h]h) \left(1 - \frac{N([t/h]h)}{K}\right) \\ &\quad \times (N([t/h]h) - m) - q N([t/h]h), \end{aligned} $$

with initial condition $N(0) = N_0$. Let $t \in [0, h)$, $t/h \in [0, 1)$, then we have

$$ {}^{C}\mathcal{D}_t^{\alpha} N(t) = r N_0 \left(1 - \frac{N_0}{K}\right)(N_0 - m) - q N_0. \tag{6}$$

The solution of Eq. 6 is

$$ N_1 = N_0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} \left[r N_0 \left(1 - \frac{N_0}{K}\right)(N_0 - m) - q N_0 \right]. $$

Next, let $t \in [h, 2h)$, $t/h \in [1, 2)$. Thus, we obtain

$$ {}^{C}\mathcal{D}_t^{\alpha} N(t) = r N_1 \left(1 - \frac{N_1}{K}\right)(N_1 - m) - q N_1, \tag{7}$$

where its solution is given by

$$N_2 = N_1(h) + \frac{(t-h)^\alpha}{\Gamma(1+\alpha)} \left[rN_1 \left(1 - \frac{N_1}{K}\right)(N_1 - m) - qN_1 \right].$$

By proceeding the same disretization process, for $t \in [nh, (n+1)h)$, $t/h \in [n, n+1)$, we have

$$N_{n+1} = N_n(nh) + \frac{(t-nh)^\alpha}{\Gamma(1+\alpha)} \left[rN_n(nh) \left(1 - \frac{N_n(nh)}{K}\right)(N_n(nh) - m) - qN_n(nh) \right]. \tag{8}$$

For $t \to (n+1)h$, Eq. 8 is reduced to a discrete-time fractional-order logistic model with the Allee effect and proportional harvesting

$$N_{n+1} = N_n + \frac{h^\alpha}{\Gamma(1+\alpha)} N_n \left[r\left(1 - \frac{N_n}{K}\right)(N_n - m) - q \right] = f(N). \tag{9}$$

We remark that if $\alpha \to 1$ then Eq. 9 is exactly the same as the Euler discretization of model (5).

3 Fundamental concepts

To analyze the dynamical behavior such as the existence of fixed point, the local stability, and the occurrence of saddle-node and period-doubling bifurcation of the discrete-time model (9), the following definition and theorems are needed.

Definition 1 [46] Consider the following map

$$x(n+1) = f(x(n)). \tag{10}$$

A point x^* is said a fixed point of the map (10) if $f(x^*) = x^*$. If $\left|f'(x^*)\right| \neq 1$, then x^* is called a hyperbolic fixed point, and if $\left|f'(x^*)\right| = 1$, then x^* is called a nonhyperbolic fixed point.

Theorem 1 *[46] Let x^* be a hyperbolic fixed point of the map (10) where f is continuously differentiable at x^*. The following statements then hold true:*

(i) If $\left|f'(x^*)\right| < 1$, then x^* is locally asymptotically stable.
(ii) If $\left|f'(x^*)\right| > 1$, then x^* is unstable.

Theorem 2 *[46] Let x^* is a nonhyperbolic fixed point of the map (10) satisfying $f'(x^*) = 1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at x^*, then the following statements hold:*

(i) If $f''(x^*) \neq 0$, then x^* unstable (semistable).
(ii) If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then x^* unstable.
(iii) If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* locally asymptotically stable.

Definition 2 [46] The Schwarzian derivative, Sf, of a function f is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2.$$

Particularly, if $f'(x^*) = -1$ then

$$Sf(x^*) = -f'''(x^*) - \frac{3}{2}\left[f''(x^*)\right]^2.$$

Theorem 3 *[46] Let x^* is a hyperbolic fixed point of the map (10) satisfying $f'(x^*) = -1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at x^*, then the following statements hold:*

(i) If $Sf(x^*) < 0$, then x^* is locally asymptotically stable.
(ii) If $Sf(x^*) > 0$, then x^* is unstable.

Theorem 4 *(The existence of Saddle-Node Bifurcation [46]) Suppose that $x_{n+1} = f(\mu, x_n)$ is a C^2 one-parameter family of one-dimensional maps, and x^* is a fixed point with $f'(\mu, x) = 1$. Assume further that*

$$\frac{\partial f}{\partial \mu}(\mu^*, x^*) \neq 0 \text{ and } \frac{\partial^2 f}{\partial x^2}(\mu^*, x^*) \neq 0.$$

Then there exists an interval I around x^* and a C^2 map $\mu = p(x)$, where $p : I \to \mathbb{R}$ such that $p(x^*) = \mu^*$, and $f(p(x), x) = x$. Moreover, if $\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}|_{(\mu^*, x^*)} < 0$, the fixed points exist for $\mu > \mu^*$, and if $\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}|_{(\mu^*, x^*)} > 0$, the fixed points exist for $\mu < \mu^*$.

Theorem 5 *(The existence of period-doubling bifurcation [46]) Suppose that $x_{n+1} = f(\mu, x_n)$ is a C^2 one-parameter family of one-dimensional maps, and x^* is a fixed point. Assume that*

(i) $\frac{\partial f}{\partial x}(\mu^*, x^*) = -1$.
(ii) $\frac{\partial^2 f}{\partial \mu \partial x}(\mu^*, x^*) \neq 0$.

Then there is an interval I about x^* and a function $p : I \to \mathbb{R}$ such that $f_{p(x)}(x) \neq x$ but $f^2_{p(x)} = x$.

4 Analytical results

□

We explore some analytical results here such as the existence of fixed points, their local stability, and the existence of some bifurcations, namely saddle-node and period-doubling bifurcations. Since the map (9) is constructed by PWCA with step size (h), the analytical process is then investigated by considering the impact of h. Some analytical results also examine the influence of the harvesting (q) on the dynamics of the given map.

4.1 The existence of fixed point

Based on Definition (1), the fixed point of the map (9) is obtained by solving the following equation

$$N = N + \frac{h^\alpha}{\Gamma(1+\alpha)} N \left[r \left(1 - \frac{N}{K} \right) (N - m) - q \right]. \quad (11)$$

The solutions of Eq. 11 are described as follows:

(i) The extinction of population fixed point $N_0^* = 0$ which always exists.
(ii) The nonzero fixed points $N_{1,2}^*$ which are the positive solutions of the following quadratic polynomial

$$N^2 - (m + K)N + mK + \frac{qK}{r} = 0. \quad (12)$$

The solutions of Eq. 12 are

$$\begin{aligned} N_1^* &= \frac{m+K}{2} + \frac{\sqrt{(q^* - q)rK}}{r}, \\ N_2^* &= \frac{m+K}{2} - \frac{\sqrt{(q^* - q)rK}}{r}, \end{aligned} \quad (13)$$

where $q^* = \frac{(m-K)^2 r}{4K} > 0$. The existence of nonzero fixed points (13) is shown by Theorem 6.

Theorem 6 *(i) If $q > q^*$, then the nonzero fixed point of the map (9) does not exist.*
(ii) If $q = q^*$, then there exists a unique nonzero fixed point $N^* = \frac{m+K}{2}$ of the map (9).
(iii) If $q < q^*$, then there exist two nonzero fixed points, namely $N_{1,2}^*$ of the map (9).

Proof (i) It is easy to confirm that if $q > q^*$ then the solutions of Eq. 12 are a pair of complex conjugate numbers.
(ii) For $q = q^*$, we have $N^* = N_1^* = N_2^* = \frac{m+K}{2}$. Hence, N^* is the only positive fixed point of the map (9).
(iii) If $q < q^*$, then $N_{1,2}^* \in \mathbb{R}$. Because $N_1^* N_2^* = mK + \frac{qK}{r} > 0$ and $N_1^* + N_2^* = m + K > 0$, then N_1^* and N_2^* are obviously positive, showing that there are two nonzero fixed points.

4.2 Local stability

Now, the dynamical behaviors of the map (9) around each fixed point are investigated. Theorems 7,10 are presenting to describe the local dynamics of fixed points N_i^*, $i = 0, 1, 2$, and N^*. The complete dynamics including local stability, unstable condition, and nonhyperbolic properties for each fixed point are studied by employing Theorems 1,2,3. In this respect, all dynamical properties are expressed in step size (h) and harvesting rate (q) to simplify the mathematical terms.

Theorem 7 *Let's denote $\hat{q} = \frac{(2m^2+3mK+2K^2)r}{K}$ and $h_0 = \sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{mr+q}}$. Then the following statements hold:*

(i) if $0 < h < h_0$, then N_0^* is locally asymptotically stable,
(ii) if $h > h_0$, then N_0^* is unstable, and
(iii) if $h = h_0$, then N_0^* is nonhyperbolic fixed point. Furthermore, if

(iii.a) $q < \hat{q}$, then N_0^* is locally asymptotically stable, and
(iii.b) $q > \hat{q}$, then N_0^* is unstable.

Proof By evaluating $f'(N)$ at N_0^*, we obtain

$$f'(N_0^*) = 1 - \frac{h^\alpha (mr + q)}{\Gamma(1+\alpha)} = 1 - 2\left(\frac{h}{h_0}\right)^\alpha.$$

(i) If $0 < h < h_0$, then $0 < (h/h_0)^\alpha < 1$, which implies $\left| f'(N_0^*) \right| < 1$. Based on Theorem 1, we have a locally asymptotically stable N_0^*.
(ii) If $h > h_0$, then $(h/h_0)^\alpha > 1$, so that $f'(N_0^*) < -1$. Theorem 1 states that N_0^* is an unstable fixed point.
(iii) For $h = h_0$, we have $f'(N_0^*) = -1$, i.e., N_0^* is nonhyperbolic fixed point. The Schwarzian derivative of map $f(N)$ at N_0^* is

$$\begin{aligned} Sf(N_0^*) =& \frac{h^\alpha}{\Gamma(1+\alpha)} \left[\frac{6r}{K} \right] \\ &- \frac{3}{2} \left[\frac{h^\alpha}{\Gamma(1+\alpha)} \left[\frac{2(m+K)r}{K} \right] \right]^2 \\ =& \frac{6rh^\alpha}{K\Gamma(1+\alpha)} \left[1 - \frac{h^\alpha}{K\Gamma(1+\alpha)} (m+K)^2 r \right] \\ =& \frac{12r}{(mr+q)K} \left[1 - \frac{2(m+K)^2 r}{(mr+q)K} \right]. \end{aligned}$$

If $q < \hat{q}$, then $Sf(N_0^*) < 0$, and thus, N_0^* is locally asymptotically stable. On the contrary, if $q > \hat{q}$ then $Sf(N_0^*) > 0$, showing N_0^* is unstable. Thus, Theorem 7 is completely proved.

□

Theorem 8 *The nonzero fixed point N^* is semistable.*

Proof The derivative of $f(N)$ at N^* is

$$f'(N^*) = 1 + \frac{h^\alpha}{\Gamma(1+\alpha)} \left[-\frac{3r}{K}\left(\frac{m+K}{2}\right)^2 + \frac{2r(m+K)}{K}\left(\frac{m+K}{2}\right) - mr - q \right]$$
$$= 1 + \frac{h^\alpha}{\Gamma(1+\alpha)} \left[\frac{(m+K)^2 r - 4mrK}{4K} - q \right]$$
$$= 1 + \frac{h^\alpha}{\Gamma(1+\alpha)} \left[q^* - q \right].$$

N^* exists when $q = q^*$. Clearly that $f'(N^*) = 1$, and therefore, N^* is a nonhyperbolic fixed point. By direct calculations, we can show that

$$f''(N^*) = \frac{h^\alpha}{\Gamma(1+\alpha)} \left[-\frac{6rN}{K} + 2\left(1 + \frac{m}{K}\right) r \right]$$
$$= \frac{h^\alpha}{\Gamma(1+\alpha)} \left[-\frac{6r}{K}\frac{m+K}{2} + \frac{2(m+K)r}{K} \right]$$
$$= \frac{h^\alpha}{\Gamma(1+\alpha)} \left[-\frac{3(m+K)r}{K} + \frac{2(m+K)r}{K} \right]$$
$$= -\frac{(m+K)rh^\alpha}{K\Gamma(1+\alpha)} < 0.$$

Since $f''(N^*) \neq 0$, Theorem 2 says that the fixed point N^* is semistable. □

Theorem 9 *Suppose that:*

$$h_1 = \sqrt[\alpha]{\frac{2K\Gamma(1+\alpha)}{2(q^*-q)K + (m+K)\sqrt{(q^*-q)rK}}},$$
$$\hat{h} = \frac{(m+K)r + 6\sqrt{(q^*-q)K}}{\sqrt{4(q^*-q)rK + 2r(m+K)\sqrt{(q^*-q)rK}}}.$$

The local stability of N_1^* is described as follows.

(i) If $0 < h < h_1$, then N_1^* is locally asymptotically stable.
(ii) If $h > h_1$, then N_1^* is unstable.
(iii) If $h = h_1$ and

(iii.a) if $\hat{h} > 1$, then N_1^* is locally asymptotically stable, and
(iii.b) if $\hat{h} < 1$, then N_1^* is unstable.

Proof It is obvious to show that

$$f'(N_1^*) = 1 + \tfrac{h^\alpha}{\Gamma(1+\alpha)} \Big[-\Big(\tfrac{3}{4K}(m+K)^2 r + 3(q^*-q) + \tfrac{3}{K}(m+K)\sqrt{(q^*-q)rK} \Big) + \tfrac{1}{K}(m+K)^2 r + \tfrac{2}{K}(m+K)\sqrt{(q^*-q)rK} - (mr+q) \Big]$$
$$= 1 - \tfrac{h^\alpha}{K\Gamma(1+\alpha)} \left[2(q^*-q)K + (m+K)\sqrt{(q^*-q)rK} \right]$$
$$= 1 - \tfrac{h^\alpha}{K\Gamma(1+\alpha)} \left[\tfrac{2K\Gamma(1+\alpha)}{h_1^\alpha} \right]$$
$$= 1 - 2\left(\tfrac{h}{h_1}\right)^\alpha.$$

Hence, we have the following observations:

(i) For $0 < h < h_1$, we have $\left|f'(N_1^*)\right| < 1$. According to Theorem 1, the nonzero fixed point N_1^* is locally asymptotically stable.
(ii) If $h > h_1$, then we get $f'(N_1^*) < -1$. Thus, N_1^* is an unstable fixed point (see to Theorem 1).
(iii) Clearly that $f'(N_1^*) = -1$ whenever $h = h_1$, which shows that N_1^* is nonhyperbolic fixed point. The Schwarzian derivative of $f(N)$ at N_1^* is given by

$$Sf(N_1^*) = \tfrac{h^\alpha}{\Gamma(1+\alpha)} \left[\tfrac{6r^2}{K} \right] - \tfrac{3}{2} \left[-\tfrac{h^\alpha}{\Gamma(1+\alpha)} \left[\tfrac{(m+K)r}{K} + \tfrac{6}{K}\sqrt{(q^*-q)K} \right] \right]^2$$
$$= \tfrac{rh^\alpha}{K\Gamma(1+\alpha)} \left[6r - \tfrac{3}{2K} \left[\tfrac{h^\alpha}{\Gamma(1+\alpha)} \right] [(m+K)r + 6\sqrt{(q^*-q)K}]^2 \right].$$

We can easily check that if $\hat{h} > 1$ then $Sf(N_1^*) < 1$ and if $\hat{h} < 1$ then $Sf(N_1^*) > 1$. Therefore, the stability of the nonhyperbolic fixed point is explained. Finally, all of the stability conditions of fixed point N_1^* are completely determined.

□

Theorem 10 *The nonzero fixed point $N_2^* = \frac{m+K}{2} - \frac{\sqrt{(q^*-q)rK}}{r}$ is always unstable.*

Proof To investigate the stability of N_2^*, we evaluate $f'(N)$ at N_2^*:

$$f'(N_2^*) = 1 + \frac{(q^*-q)h^\alpha}{\Gamma(1+\alpha)} \left[\frac{(m+K)\sqrt{r}}{\sqrt{(q^*-q)K}} - 2 \right].$$

By simple algebraic manipulations, we can show that $\frac{(m+K)\sqrt{r}}{\sqrt{(q^*-q)K}} > 2$. Thus, $f'(N_2^*)$ is always a positive constant, which means N_2^* is always an unstable fixed point. □

4.3 Bifurcation analysis

From the previous analysis, we have a nonhyperbolic fixed point N^* when $q = q^*$, indicating the possibility of the occurrence of saddle-node bifurcation. Moreover, the occurrence of period-doubling bifurcation is also indicated around the nonhyperbolic fixed point N_1^* when $h = h_1$. Thus, in this section, we study the existence of saddle-node and period-doubling bifurcations. The saddle-node bifurcation is a phenomenon that two fixed points with opposite signs of stability merge into a unique semistable fixed point and finally disappear when a parameter is varied, while the period-doubling bifurcation is a phenomenon that a single fixed point losses its stability accompanied by the emergence of a period-2 solution when a parameter is varied [46]. As results, we have Theorems 11, 12.

Theorem 11 *The nonzero fixed point N^* undergoes a saddle-node bifurcation when q crosses the critical values $q^* = \frac{(m-K)^2 r}{4K}$.*

Proof It was shown previously that N^* does not exist if $q > q^*$. When $q = q^*$, we have a semistable fixed point N^*; if $q < q^*$, then there exists two nonzero fixed points. By straightforward calculations, we have $\frac{\partial f(N^*)}{\partial N} = 1$, $\frac{\partial f(N^*)}{\partial q} = -\frac{h^\alpha}{\Gamma(1+\alpha)} \frac{m+K}{2} < 0$, and $\frac{\partial^2 f(N^*)}{\partial N^2} = -\frac{(m+K)rh^\alpha}{K\Gamma(1+\alpha)} < 0$. Thus, according to Theorem 4, the fixed point N^* undergoes a saddle-node bifurcation when q crosses the critical values $q^* = \frac{(m-K)^2 r}{4K}$. Moreover, the fixed points exist when $q \le q^*$ because $\frac{\partial f(N^*)}{\partial q} \frac{\partial^2 f(N^*)}{\partial N^2} > 0$. □

Theorem 12 *The nonzero fixed point N_1^* undergoes a period-doubling bifurcation when h crosses the critical value $h_1 = \sqrt[\alpha]{\frac{2K\Gamma(1+\alpha)}{2(q^*-q)K + (m+K)\sqrt{(q^*-q)rK}}}$.*

Proof From the proof of Theorem 9, we have that if $h = h_1$ then $\frac{\partial f(N_1^*)}{\partial N} = -1$. By performing some algebraic calculations, we also have

$$\begin{aligned} \frac{\partial^2 f(N_1^*)}{\partial h \partial N} &= \frac{\alpha h^{\alpha-1}}{\Gamma(1+\alpha)} \left[r \left(\frac{m+K}{2} + \frac{\sqrt{(q^*-q)rK}}{r} \right) \right. \\ &\quad \left. \left(1 - \frac{2}{K} \left(\frac{m+K}{2} + \frac{\sqrt{(q^*-q)rK}}{r} \right) + \frac{m}{K} \right) \right] \\ &= -\frac{r\alpha h^{\alpha-1}}{\Gamma(1+\alpha)} \left(\frac{m+K}{2} + \frac{\sqrt{(q^*-q)rK}}{r} \right) \\ &\quad \left(\frac{2}{K} \frac{\sqrt{(q^*-q)rK}}{r} \right). \end{aligned} \tag{14}$$

N_1^* exists if $q \ne q^*$, and thus, we have $\frac{\partial^2 f(N_1^*)}{\partial h \partial N} \ne 0$. According to Theorem 5, there appears a solution of period-2 when h passes through h_1. Hence, the occurrence of period-doubling bifurcation in the map (9) is completely proved. □

Theorem 12 states that the period-doubling bifurcation in the map (9) can be achieved by varying the step size h. However, such bifurcation can also be realized by setting a fixed value of h and other parameters while varying a certain parameter. In the following section, we give an example of period-doubling bifurcation which is driven by the constant of harvesting (q).

5 Numerical results

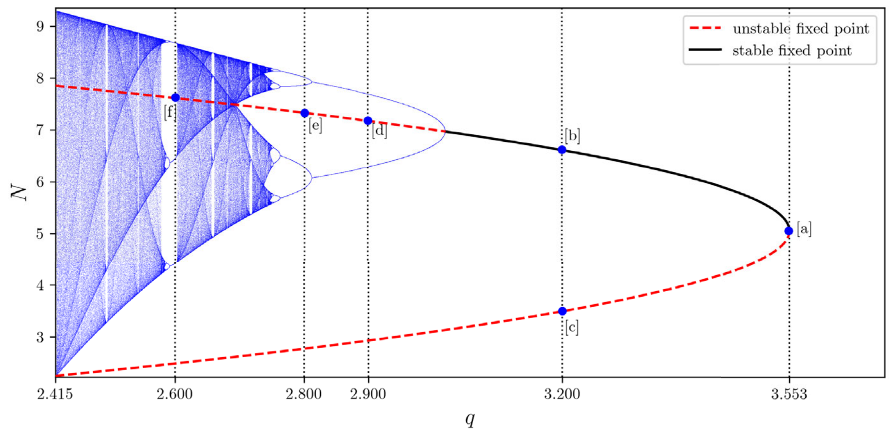
In this section, we present some numerical simulations of the map (9) not only to support the previous analytical findings but also to show more dynamical behaviors of the map (9). Numerical simulations are given by considering some biological and mathematical aspects such as the influence of the harvesting, the step size (h), the Allee effect (m), and the order α. To support the numerical simulations, a desktop PC is used based on AMD Ryzen 5 3400 G 3.7GHz, 16 GB RAM, and AMD Radeon RX580 8GB DDR5 VGA card. We also use an open-source software called Python 3.9 to generate all of the given figures. Due to the field data limitation, we use hypothetical parameter values for the numerical simulations. General parameter values are given as follows.

$$\begin{aligned} &r = 1.45,\ K = 10,\ m = 0.1,\ q = 0.32,\ \alpha = 0.8, \\ &\text{and } h = 0.4. \end{aligned} \tag{15}$$

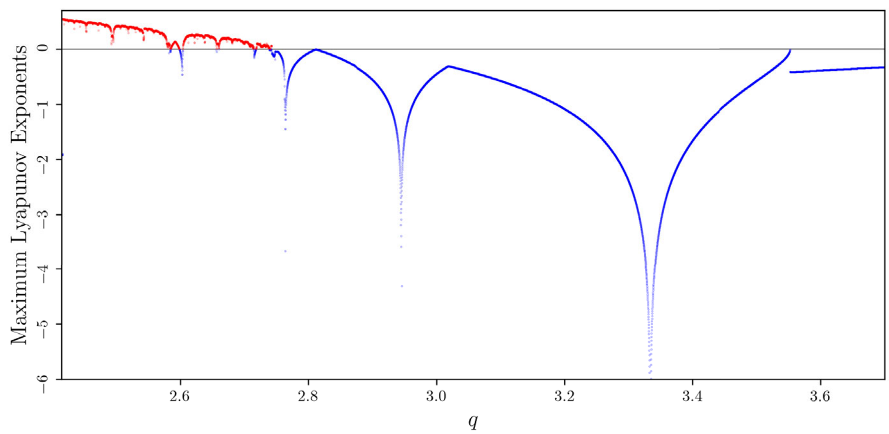
5.1 The influence of the Harvesting Rate

The numerical simulations in this subsection are using parameter set (15) and vary the value of the harvesting rate (q). According to Theorem 6, map (9) with parameter set (15) has critical value $q^* \approx 3.5527$ such that map (9) does not have a nonzero fixed point if $q > q^*$. When $q = q^*$, map (9) has a unique nonzero fixed point $N^* = 5.05$ which is a semistable fixed point, see Theorem 8. Furthermore, if $q < q^*$, then there are two nonzero fixed points, namely N_1^* and N_2^*. By taking $h = 0.4$ and using Theorem 9, we can show that N_1^* is asymptotically stable if $q_1 = 3.0191 \lesssim q < q^*$. On the other hand, Theorem 10 states that N_2^* is always unstable. Since we take $h = 0.4$, Theorem 12 states that the fixed point N_1^* undergoes a period-doubling bifurcation when q crosses q_1 from the right. To see these dynamical behaviors, we plot in Fig. 1a the bifurcation diagram of the map 9 with parameter set (15) and $h = 0.4$ for $2.415 \le q \le 3.7$. Clearly that this bifurcation diagram fits perfectly with the results of our previous analysis. Indeed, Fig. 1a shows that N^* (labeled as [a]) is semistable, see also the Cobweb diagram shown in Fig. 2a. As the value of q decreases from q^*, the nonzero fixed point is split into two nonzero fixed points where one of them is

Fig. 1 Bifurcation diagram and its corresponding maximum Lyapunov exponents of the map (9) with parameter set (15) and $2.415 \le q \le 3.7$



(a) Bifurcation diagram



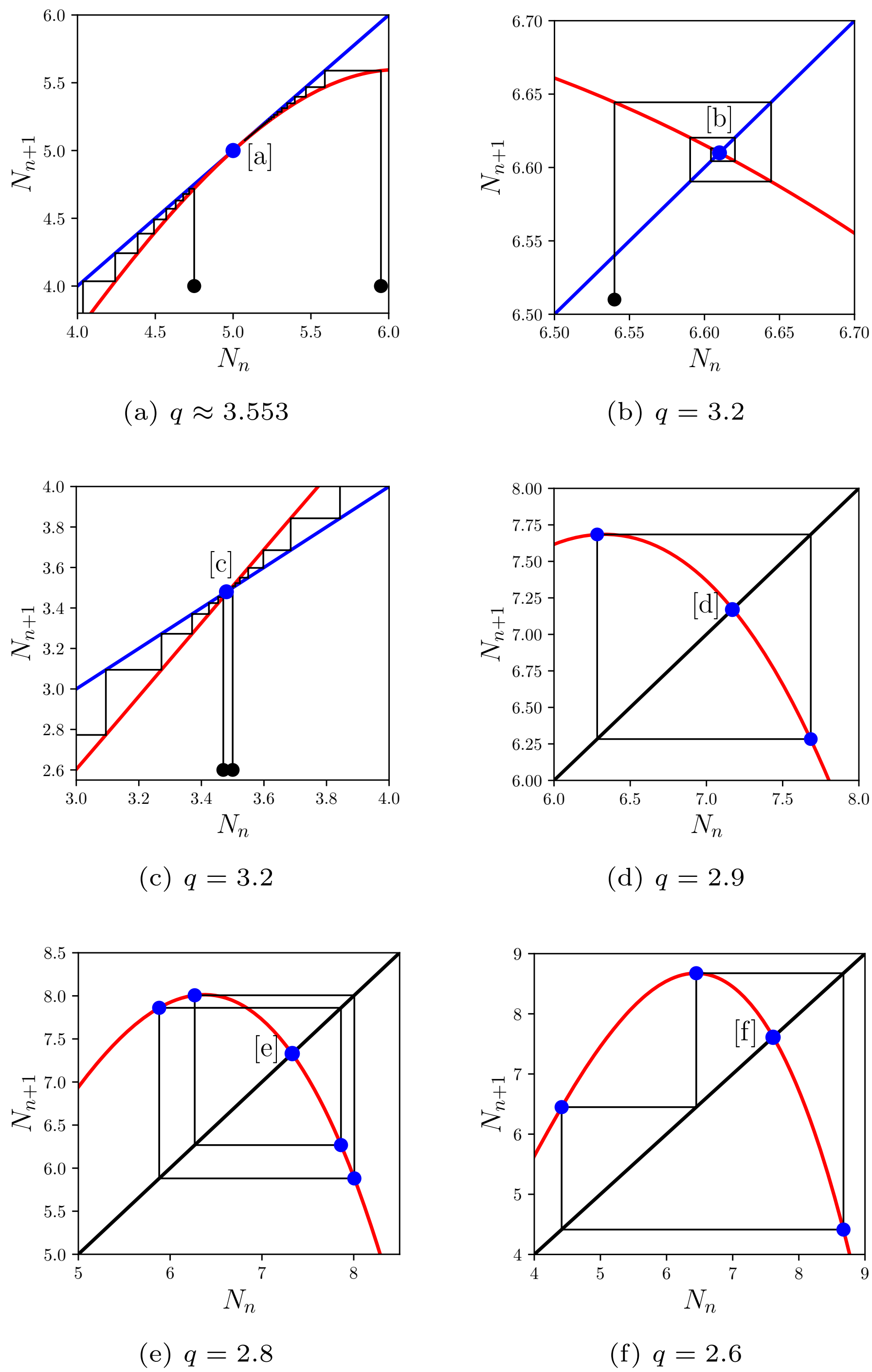
(b) Maximum Lyapunov exponents

stable in the specified interval of q, while the other fixed point is unstable. Such stability properties can also be seen in the Cobweb diagrams in Fig. 2b, c, which corresponds to points [b] and [c] in Fig. 1a, respectively. We also observe numerically the appearance of a period-doubling route to chaos (flip bifurcation) as q decreases. If we further decrease the value of q, then there appears a stable solution of period-2 when q passes through q_1. The appearance of a stable period-doubling solution, as well as a solution of period-3, is shown in Fig. 1a (see, e.g., point [d], [e], and [f], respectively, and their corresponding diagram Cobweb in Fig. 1d, e, and bif1f). The appearance of the period-3 solution indicates that our system exhibits chaotic dynamics [47]. The existence of chaotic dynamics can also be determined from the Lyapunov exponent. A system exhibits chaotic dynamics if it has positive maximum Lyapunov exponents. The maximum Lyapunov exponents which correspond to Fig. 1a are depicted in Fig. 1b. It is clearly seen that our system has positive maximum Lyapunov exponents, showing the existence of chaotic dynamics in the map (9) which is controlled by the constant of harvesting (q).

5.2 The influence of the step size

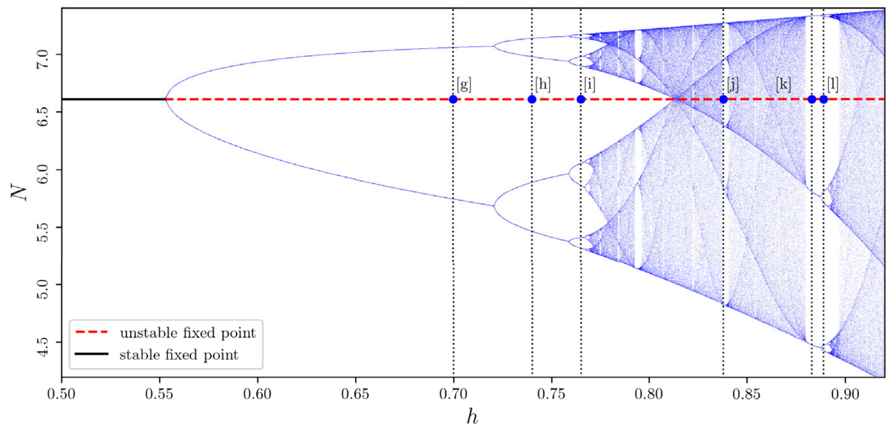
To describe the existence of period-doubling bifurcation driven by the step size h numerically, we perform simulations using the parameter set 15 and $0.5 \le h \le 0.985$. Map (9) with these parameter values has two nonzero fixed points, namely $N_1^* \approx 6.61$ and $N_2^* \approx 3.49$. N_2^* is unstable while N_1^* is stable if $0 < h < h_1 \approx 0.553$. N_1^* losses its stability via period-doubling bifurcation when h crosses h_1. These dynamics are seen in the bifurcation diagram, see Fig. 3a. Increasing the value of h may destroy the stability of N_1^*, and the system is convergent to a stable period-2 solution. Further increasing the value of h leads to a stable period-4 cycle, and so on. To give a more detailed view, we plot Cobweb diagrams in Fig. 4 which correspond to some solutions around the fixed points labeled as [g - l] in Fig. 3a. When $h = 0.7$, we have a stable period-2 cycle near the nonzero fixed point [g],

Fig. 2 Cobweb diagrams of the map (9) with parameter set (15)

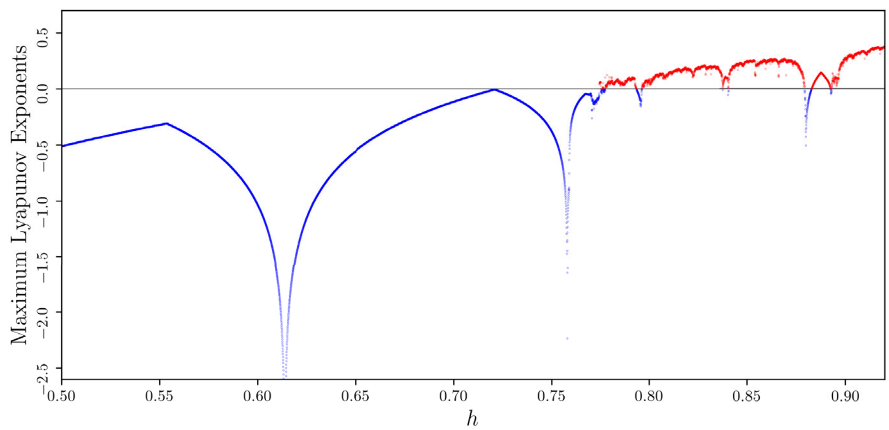


see Fig. 4a. Each of the two solutions splits into two solutions, respectively, and becomes a stable period-4 solution around fixed point [h] when $h = 0.74$ (Fig. 4b); consecutively, for $h = 0.765$ we have a stable period-8 cycle near fixed point [i], see Fig. 4c. Moreover, at $h = 0.838, 0.883, 0.889$ we have, respectively, a stable period-5 cycle around fixed point [j], a stable period-3 cycle around fixed point [k], and a stable period-6 cycle around fixed point [l], see their Cobweb diagrams in Fig. 4d, e, and f. Hence, the step size h is an important parameter that significantly affects the dynamics of the map (9). In this case, the map (9) exhibits a period-doubling bifurcation route to chaos driven by parameter h. Furthermore, the appearance of positive maximum Lyapunov exponents depicted in Fig. 3b which corresponds to the bifurcation diagram in Fig. 3a clearly shows the existence of chaotic behavior in the system.

Fig. 3 Bifurcation diagram and its corresponding maximum Lyapunov exponents of the map (9) with parameter set (15) and $0 \leq h \leq 0.92$



(a) Bifurcation diagram



(b) Maximum Lyapunov exponents

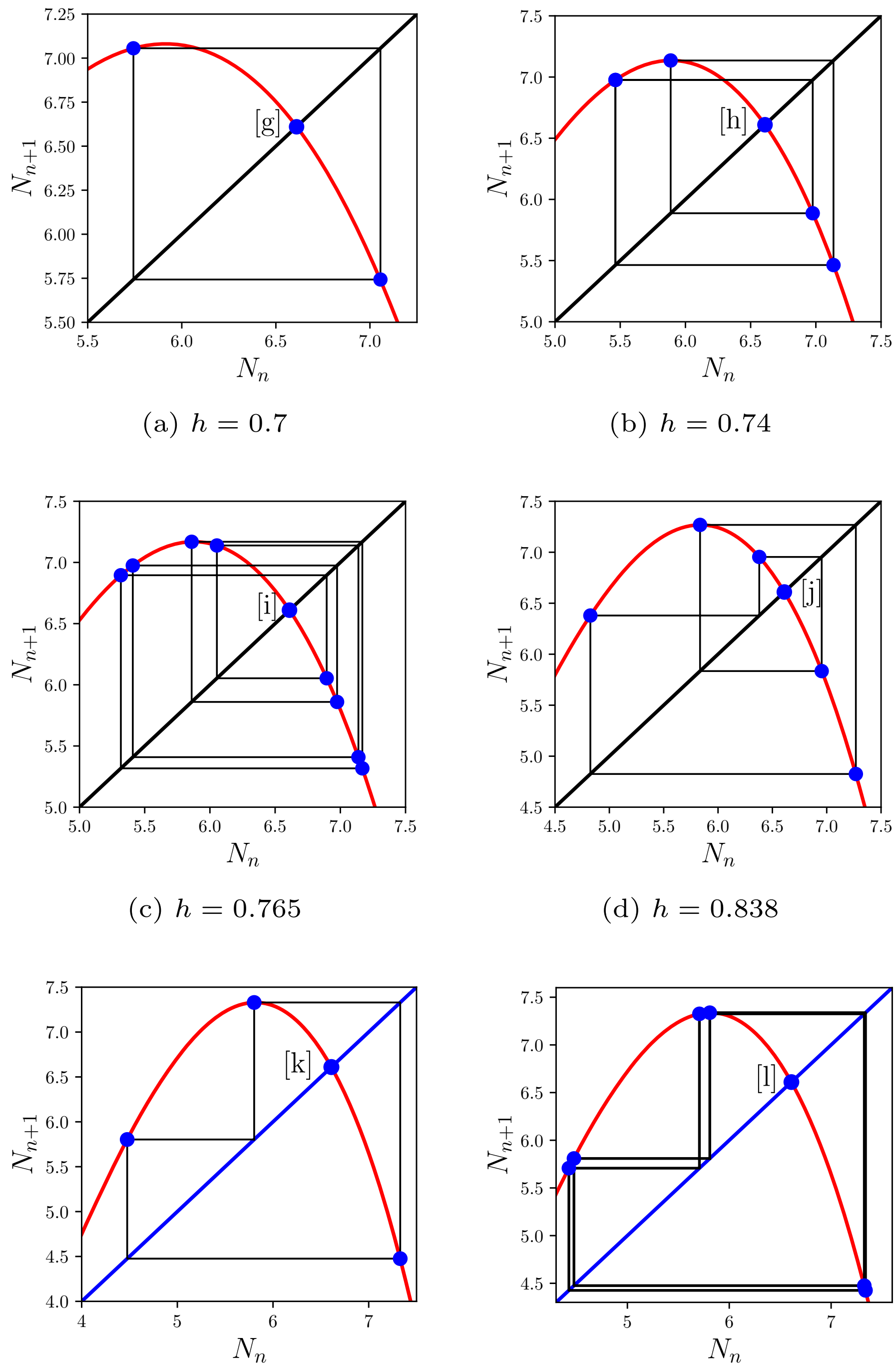
5.3 The influence of the Allee effect

To show the influence of the Allee effect, we use the parameter set (15) and vary the values of m in the interval $0 \leq m \leq 2.5$. From Eq. 13, we compute numerically that N_1^* and N_2^* exist for interval $0 \leq m \lessapprox 0.6045$. Based on Theorems 8,9,10, the stability of N_1^* and N_2^* has the different sign for $0 \leq m < 0.6045$ and finally merge into a semistable fixed point $N^* \approx 5.29671$ when $m \approx 0.6045$. When m crosses 0.6045, N^* disappears and N_0^* becomes the only fixed point of the map (9). These phenomena indicate the occurrence of saddle-node bifurcation driven by the Allee effect (m). According to Theorems 7, we also have that N_0^* is locally asymptotically stable for $m < 0.467$ and losses its stability via period-doubling bifurcation when m crosses 0.467. These complex dynamics are shown in Fig. 5a and its corresponding maximum Lyapunov exponents are depicted in Fig. 5b which confirms the existence of chaotic behavior on the map (9). One interesting condition is also shown for some values of m. For $0 < m < 0.467$, the map (9) passes through a bistability condition. N_0^* and N_1^* are locally asymptotically stable simultaneously, and hence, the solution of the map is sensitive to the initial value. See the Cobweb diagrams in Fig. 6. When $m = 0.3$, two nearby initial values are convergent to different fixed points. When the Allee effect increases to $m = 1$, the solution converges to a period-2 solution around N_0^*.

5.4 The influence of the order α

As the impact of the discretization process, we have a parameter α on map (9) which is derived from the order of the derivative of the continuous model as the memory effect. Again, we use the parameter set (15) and varying α. As result, we have a bifurcation diagram and maximum Lyapunov exponents depicted in Fig. 7. The given dynamics are quite similar to the impact of the step size but in different directions. If increasing h may change the dynamics of N_1^* from locally asymptotically stable to periodic solution via period-doubling bifurcation, different dynamics direction

Fig. 4 Cobweb diagrams of the map (9) with parameter set (15)



(a) $h = 0.7$ (b) $h = 0.74$

(c) $h = 0.765$ (d) $h = 0.838$

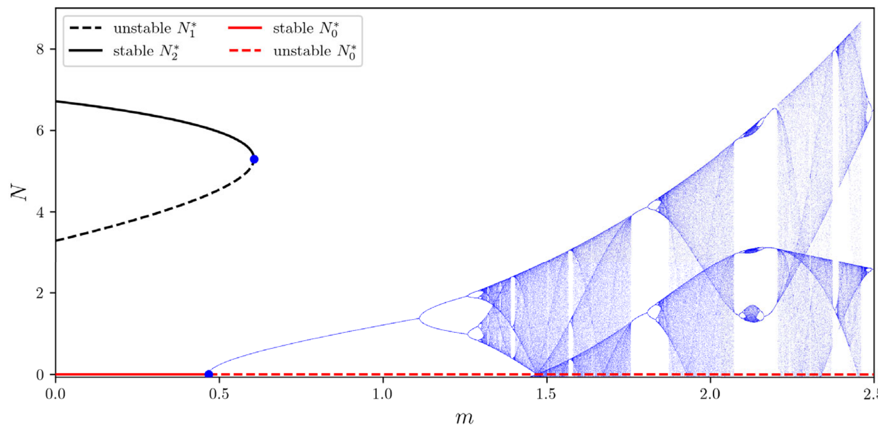
(e) $h = 0.883$ (f) $h = 0.889$

presented by α where if its value increases, the unstable N_1^* becomes locally asymptotically stable via period-doubling bifurcation. Some chaotic behavior indicated by positive Lyapunov exponents disappears becomes periodic orbits and is finally convergent to N^* when α crosses 0.5708.

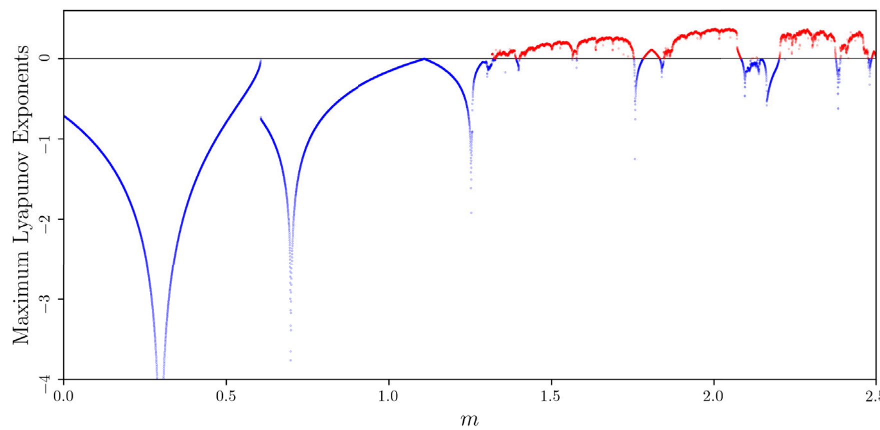
6 Hybrid control strategy

In this section, a method, namely the hybrid control strategy, is presented. This method is a combination of state feedback and parameter perturbation which is used for controlling bifurcation in a discrete system [48–51]. We first define a map

Fig. 5 Bifurcation diagram and its corresponding maximum Lyapunov exponents of the map (9) with parameter set (15) and $0 \le m \le 2.5$

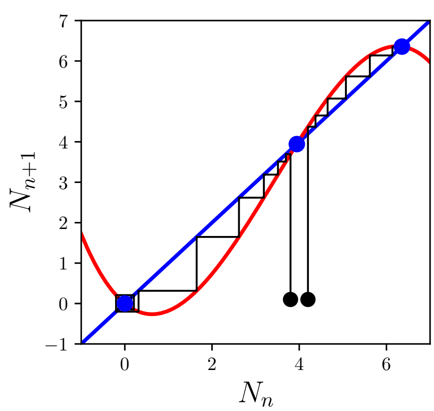


(a) Bifurcation diagram

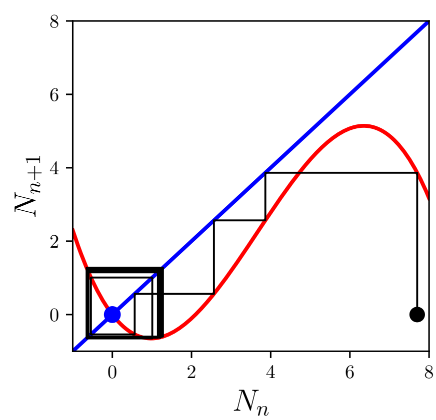


(b) Maximum Lyapunov exponents

Fig. 6 Cobweb diagrams of the map (9) with parameter set (15)

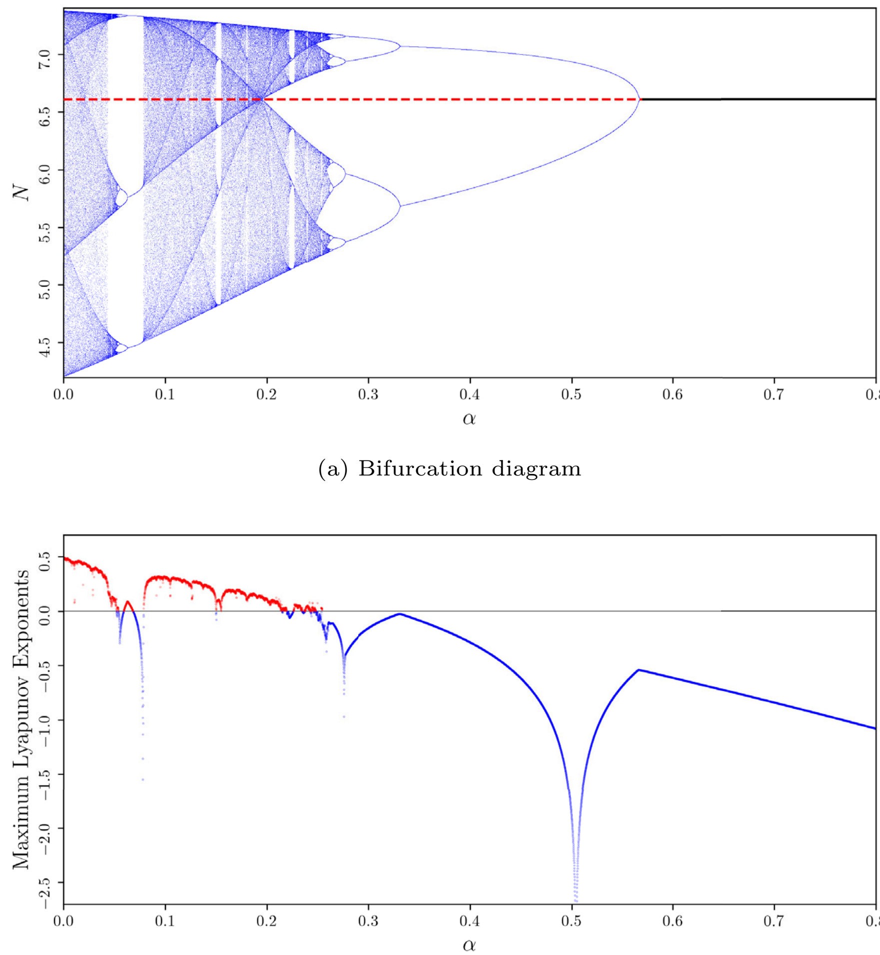


(a) $m = 0.3$



(b) $m = 1$

Fig. 7 Bifurcation diagram and its corresponding maximum Lyapunov exponents of the map (9) with parameter set (15) and $0 \le \alpha \le 0.8$



(a) Bifurcation diagram

(b) Maximum Lyapunov exponents

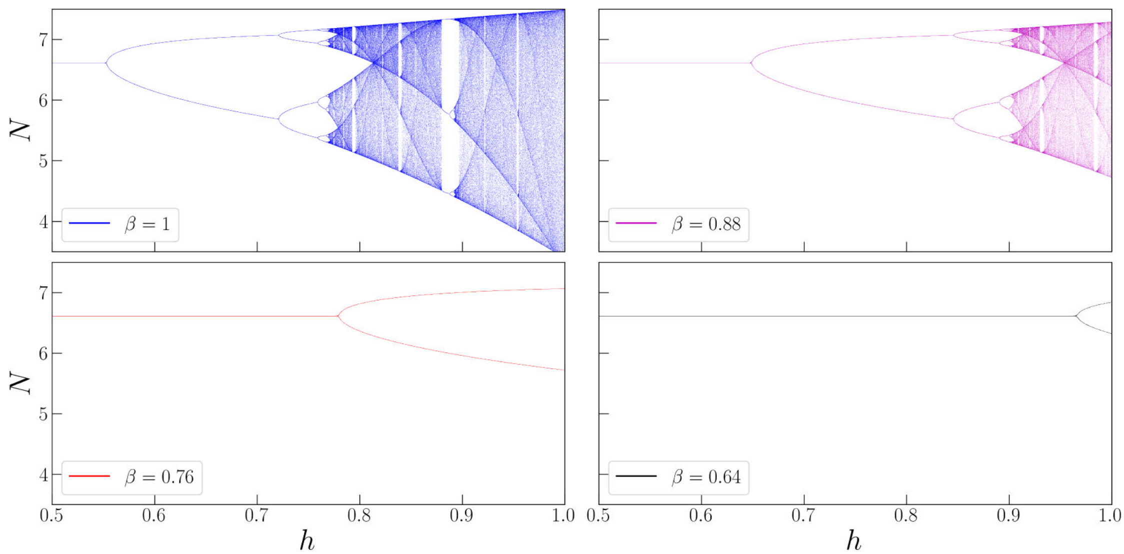
(9) as follows.

$$N_{n+1} = f(N_n, \zeta), \tag{16}$$

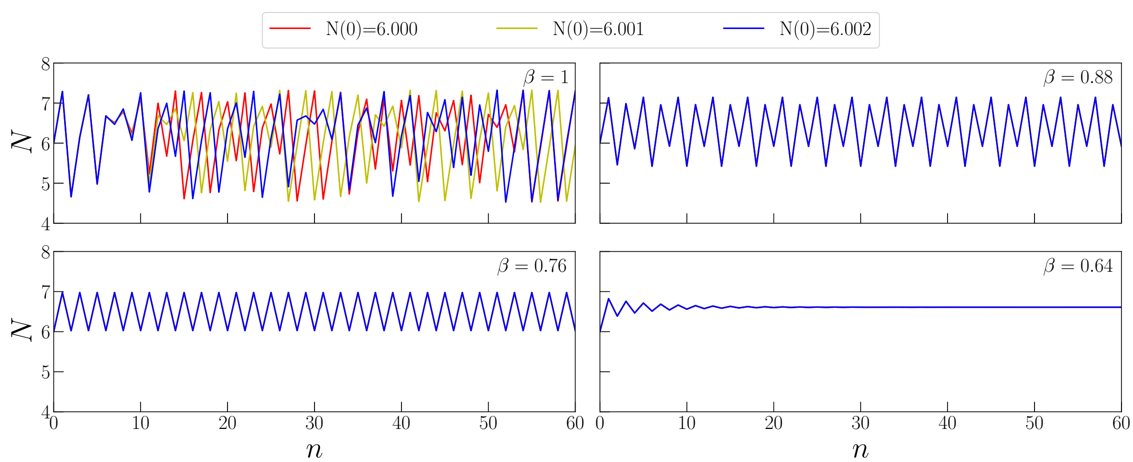
where $N \in \mathbb{R}$ is the population density and $F(N_n, \zeta)$ is the right-hand side of map (9) with bifurcation parameter $\zeta \in \mathbb{R}$. It can be revisited from analytical and numerical results that when h and q are varies in some range, the map (9) passes through a series of period-doubling bifurcations where the route to chaos. By obeying state feedback and parameter perturbation to the map (9), we obtain the control map as follows.

$$N_{n+1} = \beta f(N_n, \zeta) + (1-\beta)N_n = F(N, \beta), \tag{17}$$

where $\beta \in [0, 1]$ denotes the external control parameter for map (17). We can easily show that the map (9) and (17) have similar fixed points. From Theorem 12, N_1^* is the fixed point which undergoes a period-doubling bifurcation. Particularly, from Theorem 1 in [51], the m-periodic orbit of control map (17) is also similar to the original map (9). Now, we will show that by setting β and varying h, the occurrence of period-doubling bifurcation can be delayed or even eliminated. From the control map (17), we have $F'(N_1^*) = 1 - 2\beta\left(\frac{h}{h_1}\right)^{\alpha}$ and $\frac{\partial^2 F(N_1^*)}{\partial h \partial N} = \frac{\partial^2 f(N_1^*)}{\partial h \partial N} < 0$. According to Theorem 5, the control map (17) also undergoes period-doubling bifurcation for the similar fixed point with map (9). The difference lies in the bifurcation point where the map (9) is $h = h_1$ while the control map (17) is $h = \frac{h_1}{\sqrt[\alpha]{\beta}}$. This means if β decreases then the bifurcation point increase which means the series of periodic solutions are delayed. For example, by setting the parameter values as in Eq. 15 and $\beta = 0.64, 0.76, 0.88, 1$, the occurrence of bifurcation is delayed and period-3 solutions disappear. See Fig. 8a. We also check the chaotic solution near the period-3 solution. For $h = 0.887$, three quite close initial conditions $N(0) = 6, 6.001, 6.002$ is given and portray the solutions in Fig. 8b. The chaotic interval which occurs for $\beta = 1$ becomes a periodic solution for $\beta = 0.76, 0.88$, and finally, converges to N_1^* when $\beta = 0.64$.



(a) Bifurcation diagram



(b) Solutions for $h = 0.877$

Fig. 8 Bifurcation diagrams of controlled map (17)

7 Conclusion

A discrete-time fractional-order logistic model with the Allee effect and proportional harvesting has been constructed and investigated dynamically. The discrete-time model is derived by applying the PWCA method to the Caputo fractional-order modified logistic model. The local stability for each fixed point is successfully investigated completely for hyperbolic and nonhyperbolic fixed points by obeying the stability theorem along with the Schwarzian derivative. Furthermore, it was shown analytically that the obtained discrete-time model exhibits a saddle-node bifurcation as well as period-doubling bifurcation. The key parameter in such bifurcations is the constant of harvesting (q) or the step size (h). Numerical simulations with varying parameters q and h confirm our analytical results. The dynamics of the map are also studied numerically by varying the Allee threshold (m) and the order α which also give the saddle-node and period-doubling bifurcations. Furthermore, the presented numerical results also showed the existence of period-doubling route chaos which is indicated by the positive Lyapunov exponents and the appearance of period-3 window. We then construct the control based on the hybrid control strategy method. It is shown that the occurrence of period-doubling can be delayed. The occurrence of the chaotic solution is also successfully eliminated when the control parameter is decreased.

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Code availability Not applicable.

Declarations

Conflict of interest All authors declare that they have no conflict of interest.

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