Stability Analysis of a Fractional-Order Leslie-Gower Model with Allee Effect in Predator

by Hasan S. Panigoro

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Abstract. In this paper, the dynamics of a fractional-order Leslie-Gower model with Allee effect in predator is investigated. Firstly, we determine the existing condition and local stability of all possible equilibrium points. The model has four equilibrium points, namely both prey and predator extinction point, the prey extinction point, the predator extinction point, and the interior point. Furthermore, we also show the dynamic changing around the interior point due to the changing of the order of the fractional derivative, namely the Hopf bifurcation. In the end, some numerical simulations are demonstrated to illustrate the dynamics of the model. Here we show numerically the local stability, the occurrence of Hopf bifurcation, and the impact of the Allee effect to the prey and predator densities.

1. Introduction

In 1931, an American ecologist, W. C. Allee [1] introduced a biological phenomenon, namely the Allee effect. It is referred to a condition at lower population density, the per capita growth rate and population density correlate positively. Allee effect occurs when species depends on cooperative behaviour among conspecifics during foraging, environmental conditioning, anti-predator tactics, breeding, or locating and mate finding problem at low-density [2]. In the predator-prey model, the Allee effect may occur in prey population [3, 4, 5, 6, 7], predator population [8, 9], and both population [10]. Because the predator population is more prone than their prey [8], we are interested to observe the influence of Allee effect in predator population. For example, the spotted owls (*Strix occidentalis*) with mating limitation behavior due to its habitat loss [11] and the African wild dogs (*Lycaon pictus*) with cooperative hunting behavior during their foraging activities [12]. We adopt the modified Leslie-Gower model proposed by Feng and Kang [10] and assume that the Allee effect exists only in predator with the predation behavior following Beddington-DeAngelis functional response [13]. The model is given as follows:

$$\begin{aligned}
\frac{dN}{dt} &= rN\left(1 - \frac{N}{K}\right) - \frac{bNP}{1 + cN + qP}, \\
\frac{dP}{dt} &= sP\left(\frac{P}{P + n} - \frac{eP}{k + N}\right),
\end{aligned} \qquad (1)$$

where $N(t)$ and $P(t)$ are the density of prey and predator, respectively, at time t. All parameters in the model (1) are real positive constants. In particular, parameter r is the intrinsic growth rate

prey, K is the carrying capacity of prey, q is the magnitude of interference among predators, s is the intrinsic growth rate of the predator, n is the Allee effect constant, e is the maximum value which per capita reduction rate of the predator can attain, k is the environment protection to the predator, and b and c measures the effect of capture rate and handling time by the predator to the predation rate, respectively. The term $\frac{P}{P+n}$ denotes the Allee effect function. This term is also known as the weak Allee effect [9], since at low density, the per capita growth rate is lower than at higher density but remains positive. $\frac{bNP}{1+cN+qP}$ represents the Beddington-DeAngelis functional response [3].

In this paper, the dynamics of model (1) at fractional-order is investigated. The reason to use the fractional-order instead of the integer order derivative is because in many situation the growth rates of prey and predator depend not only on the current state but also on the history of variable or its memory effect [4, 14, 15, 16]. There are several definitions for fractional-order derivative, but we are interesting in applying the Caputo fractional derivative since its convenient in application. By replacing the left hand sides of the model (1) with the Caputo fractional-order derivative, we obtain

$$
\begin{aligned}
D_*^\alpha N &= rN\left(1-\frac{N}{K}\right) - \frac{bNP}{1+cN+qP}, \\
D_*^\alpha P &= sP\left(\frac{P}{P+n} - \frac{eP}{k+N}\right),
\end{aligned}
\tag{2}
$$

where D_*^α represents the Caputo fractional-order derivative of a real valued function f which is defined by

$$
D_*^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha}\,d\tau,
$$

with $\Gamma(\cdot)$ is Gamma function and $\alpha \in (0,1]$ [17].

In order to simplify model (2), we introduce the following nondimensional model:

$$
\begin{aligned}
D_*^\alpha x &= x(1-x) - \frac{xy}{\omega + \beta x + \delta y}, \\
D_*^\alpha y &= \sigma y\left(\frac{y}{y+\theta} - \frac{y}{\eta+x}\right),
\end{aligned}
\tag{3}
$$

where $x = \frac{N}{K}, y = \frac{eP}{K}, \tau = rt, \omega = \frac{er}{bK}, \beta = \frac{cer}{b}, \delta = \frac{qer}{b}, \sigma = \frac{s}{r}, \theta = \frac{en}{K}$, and $\eta = \frac{k}{K}$.

This paper is organized as follows. In Section 2, the existence of equilibrium points and their local stability are investigated. The occurrence of Hopf bifurcation driven by the order of the fractional derivative is studied in Section 3. In Section 4, some numerical simulations are presented to validate analytical results such as the local stability, the Hopf bifurcation, and the influence of Allee effect. The paper ends with several conclusions.

2. Equilibrium Point and Local Stability
To explore the dynamical behavior, we need the following theorem.

Theorem 1. *(Matignon condition [18]) Consider a Caputo fractional-order system with initial value*

$$
D_*^\alpha \vec{x}(t) = \vec{f}(t,\vec{x}), \quad \vec{x}(0) = \vec{x}_0,
\tag{4}
$$

where $x \in \mathbb{R}^n$ and $0 < \alpha < 1$. If $\vec{x}^*$ satisfies $\vec{f}(t,\vec{x}^*) = \vec{0}$, then $\vec{x}^*$ is an equilibrium point. If all eigenvalues λ_i, $i = 1,\dots,n$ of the Jacobian matrix $J(\vec{x}^*)$ satisfy $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$ then the equilibrium points $\vec{x}^*$ is locally asymptotically stable.

Based on Theorem 1, the equilibrium point of the model (3) is obtained by solving $D_*^\alpha x = D_*^\alpha y = 0$. Thus we have

(1) Both prey and predator extinction point $E_0 = (0,0)$ which is always exists,

(2) The prey extinction point $E_1 = (0, \eta - \theta)$ which is exists if $\eta > \theta$,

(3) The predator extinction point $E_2 = (1, 0)$ which is always exists, and

(4) The interior point $E^* = (x^*, y^*)$ where $y^* = x^* + \eta - \theta$, $x^* > \theta - \eta$, and x^* are all positive roots of the quadratic equation

$$x^2 + ax - \left(a + 1 + \frac{\theta - \eta - 1}{\beta + \delta}\right) = 0, \quad a = \frac{(\omega + \delta(\eta - \theta)) - (\beta + \delta - 1)}{\beta + \delta}. \tag{5}$$

The existence of interior point E^* for model (3) is established by the following theorem.

Theorem 2. *Let $x^* > \theta - \eta$, then the following statements hold.*

(1) If $(a + 2)^2 < 4\left(\frac{1 + \eta - \theta}{\beta + \delta}\right)$, then there is no positive interior point.

(2) If $(a + 2)^2 = 4\left(\frac{1 + \eta - \theta}{\beta + \delta}\right)$ and

 (i) If $\theta < \frac{\omega + \delta\eta + 1 - (\beta + \delta)}{\delta}$, then there is no positive interior point.

 (ii) If $\theta > \frac{\omega + \delta\eta + 1 - (\beta + \delta)}{\delta}$, then there exists a positive interior point.

(3) If $(a + 2)^2 > 4\left(\frac{1 + \eta - \theta}{\beta + \delta}\right)$ and

 (i) If $\theta > \frac{\omega + \delta\eta + 1 - (\beta + \delta)}{\delta}$, then there exists a positive interior point.

 (ii) If $\theta < \min\left\{(1 + \eta) - (1 + a)(\beta - \delta), \frac{\omega + \delta\eta + 1 - (\beta + \delta)}{\delta}\right\}$, then there exist two positive interior point.

Proof. Since $x^* > \theta - \eta$, y^* is always positive. Furthermore, from eq. (5) we have

$$x_{1,2}^* = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4\left(\frac{1 + \eta - \theta}{\beta + \delta}\right)}}{2}. \tag{6}$$

(1) Let $(a + 2)^2 < 4\left(\frac{1 + \eta - \theta}{\beta + \delta}\right)$. Thus eq. (6) are a pair of complex conjugate numbers, which means E^* is not exist.

(2) When $(a + 2)^2 = 4\left(\frac{1 + \eta - \theta}{\beta + \delta}\right)$, we have $x_1^* = x_2^* = -\frac{a}{2}$. If $\theta < \frac{\omega + \delta\eta + 1 - (\beta + \delta)}{\delta}$ then $a > 0$. Hence, $x_{1,2}^*$ are not biologically feasible. Otherwise, if $\theta > \frac{\omega + \delta\eta + 1 - (\beta + \delta)}{\delta}$ we have $x_{1,2}^* \in \mathbb{R}^+$. Therefore, $E^* = (-\frac{a}{2}, -\frac{a}{2} + \eta - \theta)$ is unique equilibrium point in interior.

(3) If $(a + 2)^2 > 4\left(\frac{1 + \eta - \theta}{\beta + \delta}\right)$, then $x_{1,2}^* \in \mathbb{R}$. Furthermore, if $\theta > \frac{\omega + \delta\eta + 1 - (\beta + \delta)}{\delta}$, then $x_1^* x_2^* < 0$ which means x_1^* and x_2^* have different sign. Therefore, $x_1^* = -\frac{a}{2} + \frac{\sqrt{a^2 - 4\left(\frac{1 + \eta - \theta}{\beta + \delta}\right)}}{2} > 0$ and E^* is the only interior point. On the other hand, if $\theta < \min\left\{(1 + \eta) - (1 + a)(\beta - \delta), \frac{\omega + \delta\eta + 1 - (\beta + \delta)}{\delta}\right\}$ then $x_1^* + x_2^* > 0$ and $x_1^* x_2^* > 0$, and hence $x_{1,2}^* \in \mathbb{R}^+$. Thus, E_1^* and E_2^* are biologically equilibrium points.

The local stability analysis can be done by using the linearization technique around each equilibrium point. The Jacobian matrix J of the model (3) at any point (x, y) is followed by:

$$J(x, y) = \begin{bmatrix} 1 - 2x - \dfrac{(\omega + \delta y)y}{(\omega + \beta x + \delta y)^2} & -\dfrac{x(\omega + \beta x)}{(\omega + \beta x + \delta y)^2} \\ \dfrac{\sigma y^2}{(\eta + x)^2} & \dfrac{\sigma(y + 2\theta)y}{(y + \theta)^2} - \dfrac{2\sigma(\eta + x)y}{(\eta + x)^2} \end{bmatrix} \tag{7}$$

Theorem 3. $E_0 = (0, 0)$ *is always unstable.*

Proof. By substituting E_0 to eq. (7), we obtain the Jacobian matrix,

$$J(E_0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{8}$$

Therefore, the eigenvalues of (8) are $\lambda_1 = 1$ and $\lambda_2 = 0$. Because $|\arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$, then E_0 is always unstable.

Theorem 4. *The prey extinction point* $E_1 = (0, \eta - \theta)$ *is locally asymptotically stable if* $\omega < (1 - \delta)(\eta - \theta)$.

Proof. If E_1 is subtituted to eq. (7), we have

$$J(E_1) = \begin{bmatrix} 1 - \dfrac{(\eta - \theta)}{\omega + \delta(\eta - \theta)} & 0 \\ \dfrac{\sigma(\eta - \theta)^2}{\eta^2} & -\dfrac{\sigma(\eta - \theta)^2}{\eta^2} \end{bmatrix}. \tag{9}$$

From eq. (9) we obtain the eigenvalues $\lambda_1 = 1 - \dfrac{(\eta - \theta)}{\omega + \delta(\eta - \theta)}$ and $\lambda_2 = -\dfrac{\sigma(\eta - \theta)^2}{\eta^2}$. Because $|\arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$, the stability of E_1 depends on λ_1. If $\omega < (1 - \delta)(\eta - \theta)$ then $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$, and hence E_1 is locally asymptotically stable.

Theorem 5. $E_2 = (1, 0)$ *is a non-hyperbolic point.*

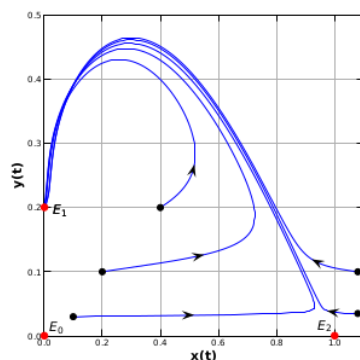*Proof.* By evaluating E_2 at eq. (7), we acquire

$$J(E_2) = \begin{bmatrix} -1 & -\dfrac{1}{\omega + \beta} \\ 0 & 0 \end{bmatrix}. \tag{10}$$

Thus, the eigenvalues of eq. (10) are $\lambda_1 = -1$ and $\lambda_2 = 0$. Since $|\arg(\lambda_2)| = \frac{\alpha\pi}{2}$, E_2 is a non-hyperbolic point.

Theorem 6. *Suppose that*

$$\chi_1 = 1 - \left[2x^* + \frac{(\omega + \delta y^*)(1 - x^*)^2}{y^*} + \sigma\left(\frac{y^*}{\eta + x^*}\right)^2\right]$$

$$\chi_2 = \frac{\sigma}{(\eta + x^*)^2}\left[\left(\omega y^* + \delta y^{*2} + \omega x^* + \beta x^{*2}\right)(1 - x^*)^2 - (1 - 2x^*)y^{*2}\right]$$

$$\alpha^* = \frac{2}{\pi}\left|\tan^{-1}\frac{\sqrt{4\chi_2 - (\chi_1)^2}}{\chi_1}\right|$$

$E^* = (x^*, y^*)$ *is locally asymptotically stable if*

4

Figure 1. Phase portrait of model (3) with parameter values: $\omega = 0.1$, $\beta = 0.8$, $\delta = 0.3$, $\sigma = 0.6$, $\theta = 0.3, \eta = 0.5$, and $\alpha = 0.9$

(i) $\chi_1^2 \geq 4\chi_2$, $\chi_1 < 0$, and $\chi_2 > 0$.

(ii) $\chi_1^2 < 4\chi_2$, and if $\chi_1 < 0$, or $\chi_1 > 0$ and $\alpha < \alpha^*$.

Proof. By using eq. (7) at interior equilibrium point $E^* = (x^*, y^*)$ we obtain

$$J(E^*) = \begin{bmatrix} 1 - 2x^* - \dfrac{(\omega + \delta y^*)(1 - x^*)^2}{y^*} & -\dfrac{x^*(\omega + \beta x^*)(1 - x^*)^2}{y^{*2}} \\ \sigma \left(\dfrac{y^*}{\eta + x^*} \right)^2 & -\sigma \left(\dfrac{y^*}{\eta + x^*} \right)^2 \end{bmatrix} \tag{11}$$

The Jacobian matrix (11) has polynomial characteristic $\lambda^2 - \chi_1 \lambda + \chi_2 = 0$. By using the Routh-Hurwitz criterion for Caputo fractional-order [19], the stability condition is completely proven.

3. Hopf Bifurcation

Hopf bifurcation on a fractional-order system occurs when a pair of complex eigenvalues belong to that system and there is a limit cycle when the stability of that system changes. According to Theorem (6), the stability of the interior equilibrium point is influence by the order of derivative when $\chi_1^2 < 4\chi_2$ and $\chi_1 > 0$. Thus, we can take the order of the derivative of a model (3) as the bifurcation parameter. Therefore, the following theorem is achieved.

Theorem 7. *(Existence of Hopf bifurcation) Let $\chi_1^2 < 4\chi_2$ and $\chi_1 > 0$. E^* undergoes a Hopf bifurcation when α crosses α^*.*

Proof. Based on Theorem 6, when $\chi_1^2 < 4\chi_2$ and $\chi_1 > 0$, the eigenvalues of model (3) at E^* are a pair of complex conjugate numbers with the real parts are positive. We also confirm that $m(\alpha^*) = 0$ and $\frac{dm(\alpha)}{d\alpha}|_{\alpha=\alpha^*} \neq 0$ where $m(\alpha) = \alpha \frac{\pi}{2} - \min_{1 \leq i \leq 2} |\arg(\lambda_i)|$. Based on Theorem 3 in [20], the equilibrium point E^* undergoes a Hopf bifurcation when α passes through α^*.

(a) $\alpha = 0.7$ (b) $\alpha = 0.74$
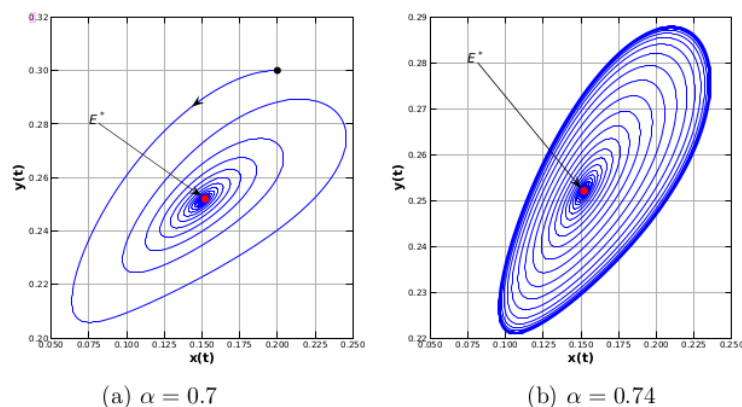
Figure 2. Phase portrait of model (3) with parameter values: $\omega = 0.1$, $\beta = 0.8$, $\delta = 0.3$, $\sigma = 0.6$, $\theta = 0.4$, and $\eta = 0.5$
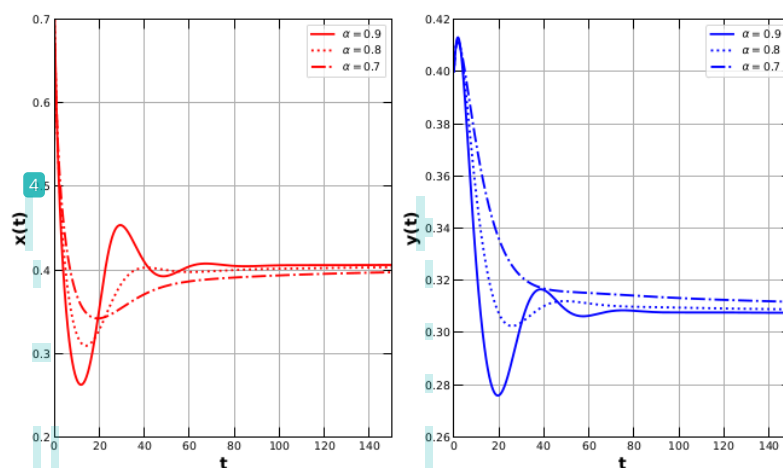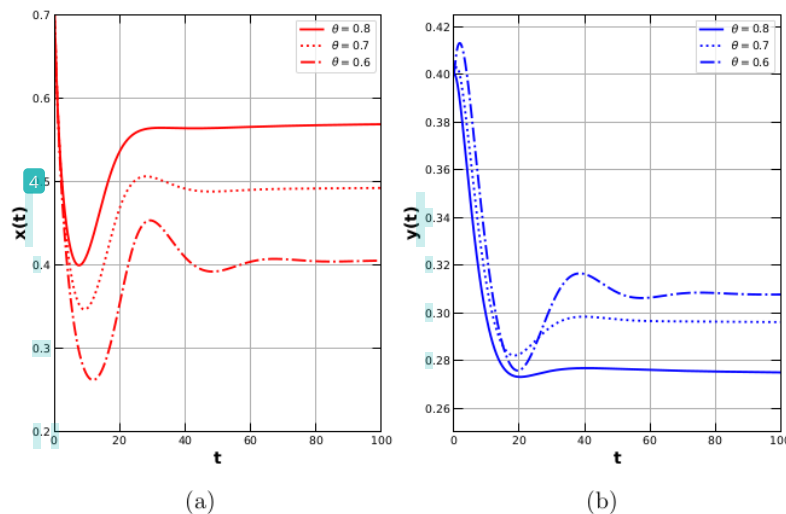

Figure 3. Time series of model (3) with parameter values: $\omega = 0.1$, $\beta = 0.8$, $\delta = 0.3$, $\sigma = 0.6$, $\theta = 0.6$, and $\eta = 0.5$

4. Numerical Simulations

Numerical simulation is given to illustrate the dynamics of model (3) which corresponds to several results in Section 2 and 3. We apply the predictor-corrector scheme for fractional-order system developed by Diethelm et al. [21]. In our work, we choose the parameter values by considering the previous stability condition. We set $\omega = 0.1$, $\beta = 0.8$, $\delta = 0.3$, $\sigma = 0.6$, $\theta = 0.3$, $\eta = 0.5$, and $\alpha = 0.9$. Therefore, we have two unstable equilibrium points $E_0 = (0,0)$ and $E_2 = (1,0)$, and a locally asymptotically stable $E_1 = (0, 0.2)$. According to Theorem 2, there is no equilibrium point in the interior of the model (3). Hence, E_1 is a unique stable equilibrium point. We give its phase portrait in Figure 1.

Next, we increases the parameter θ to 0.4. Based on Theorem 2, an equilibrium point E^*

Figure 4. Time series of model (3) with parameter values: $\omega = 0.1$, $\beta = 0.8$, $\delta = 0.3$, $\sigma = 0.6$, $\eta = 0.5$, and $\alpha = 0.6$

occurs in interior. If we set $\alpha = 0.7$, E^* becomes locally asymptotically stable as in Figure 2(a) and when $\alpha = 0.74$, E^* loses its stability as in Figure 2(b). Notice that, when $\alpha = 0.74$ the equilibrium point and nearby solutions are isolated by limit cycles and all solutions convergent to the periodic signal, see Figure 2(b). This phenomenon is called a Hopf bifurcation which corresponds to Theorem 7.

To show the influence of the order of derivative, we plot the time series of the solution by varying α. We set the parameters as follows: $\omega = 0.1$, $\beta = 0.8$, $\delta = 0.3$, $\sigma = 0.6$, $\theta = 0.6$, $\eta = 0.5$, and gives the simulation in Figure 3. The time series shows that for $\alpha = 0.7, 0.8, 0.9$, all solutions oscillates and convergent to the interior equilibrium point. It is observed that the greater the order of derivative, the faster the solution convergent towards the equilibrium point. This means the order of the derivative is directly proportional to the convergence rate.

Next, we show the impact of the Allee effect on populations density both predator and prey. We use previous parameter values, $\alpha = 0.6$ and varying the Allee effect constant θ. In Figure 4, it is shown that when θ increases, the density of predator decreases, and the density of prey increases. This means the Allee effect constant inversely proportional to the density of predator and directly proportional to the density of prey.

Conclusions

The dynamics of the fractional-order Leslie-Gower model with Allee effect in predator have been studied. We present the existence of equilibrium points and their local stability. We also show the existence of a Hopf bifurcation driven by the order of the derivative both analytically and numerically. By using numerical simulation, we show that by decreasing the order of the derivative, the slower the convergence of the solution to the equilibrium point will be. Finally, we conclude that numerically the Allee effect constant has a positive relationship with the density of prey and a negative relationship with the density of predators.

6. Acknowledgements

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