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by Hasan S. Panigoro

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A Fractional Order Predator–Prey Model with Strong Allee Effect and Michaelis–Menten Type of Predator Harvesting

Hasan S. Panigoro,^{1, 2, a)} Emli Rahmi,^{1, 2} Agus Suryanto,¹ and Isnani Darti¹

¹⁾Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Brawijaya, Indonesia ²⁾Department of Mathematics, Faculty of Mathematics and Natural Sciences, State University of Gorontalo, Indonesia

^{a)}Corresponding authors. Email: hspanigoro@ung.ac.id

Abstract. Allee effect and harvesting are two imposite of the strong Allee effect in prey and Michaelis-Menten type 49 harvesting in predators on the dynamics of a Gause-type predate 25 rey model. To involve the influence of the memory effect, the Caputo fractional-order derivative is applied. As a preliminary analysis, we obtain three types of equilibrium points namely the origin point, a pair of the 39 redator extinction points, and the co-existence point. Some interesting dynamics are shown such as 2 e local stability for each equilibrium point, the existence of transcritical bifurcatic 2 around the predator extinction points, and the co-existence point. Furthermore, some numerical simulations are performed to reinforce the theoretical findings.

Keywords: fractional-order, predator-prey, Allee effect, Michaelis-Menten harvesting

INTRODUCTION

Mathematical modeling is consistently considered a powerful scientific tool to explicate complex dynamics of prey and predator interaction [1, 2]. Various ecological components are heretofore involved in predator-prey modeling to approach the actual circumstance in nature. In this respect, we are inquisitive about the influence of two famous ecological components namely the Allee effect ar 13 havesting on the dynamics of the predator-prey model. Allee effect is introduced by W. C. Allee (1931) which proposed that the per capita growth rate declines at low population densities [3]. This phenomenon occurs as a result of several mechanisms such as social thermoregulation, mating limitation, cooperative hunting, antipredator vigilance, and so forth; see [4, 5, 6, 7] and related references therein. On the other side, harvesting exists as a consequence of human needs for biological resources 5ch as livestock and fisheries [8, 9]. In our works, we assume that the prey and predator relationship following the Gause-type predatorprey model (see [10]), the Allee effect occurs in prey using the multiplicative form as in [7], and the predator is harvested following Michaelis-Menten type [9]. For example, the prey and the predator relationship between cod (*Gadus Morhua*) and shark, where cod is exposed by Allee effect due to its difficulties in finding a mate at low population de 4 ty, and shark is hunted by humans for its fin [11, 12]. Consequently, we have a two-dimensional predator-prey model as follows.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)(N - m) - aNP,$$

$$\frac{dP}{dt} = bNP - \frac{cEP}{k_1E + k_2P} - dP,$$
(1)

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where $N(t) \ge 0$ and $P(t) \ge 0$ are prey and predator densities. r, K, a, b, c, E, and $d \ge 0$ positive constants which represent the intrinsic growth rate of prey, the en 11 pumental carrying capacity of prey. the prey capture rate by their predators, the con 45 ion efficiency from prey to predator, the catchability coefficient, the effort applied to harvest the tray species, and the death rate of the predator, respectively. k_1 and k_2 are suitable constants of the harvesting. The parameter *m* represents the Allee threshold. The term (N-m) known as the multiplicative Allee effect since its term is multiplied with the growth function $g(N) = rN\left(1 - \frac{N}{K}\right)(N-m)$ which satisfies g(0) = 0 and $g'(0) \ge 0$ whenever $m \le 0$ while g(0) = 0 and g'(0) < 0 whenever m > 0 [7]. Therefore two types of the Allee effect i.e weak Allee effect when $m \le 0$, and strong Allee effect when m > 0. In this paper, we focus to study the impact of strong Allee effect (m > 0) to the dynamics of model (1), thus the use of the words Allee effect below refers to the strong one.

To make the model (1) more easier to analyze, we perform some variables scaling i.e. x = N/K, y = aP/rK, and $\tilde{t} = rKt$. This process is to simplify model (1) by maintaining its dynamical behaviors qualitatively. Therefore, we

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achieve a nondimensional model as follows.

$$\frac{dx}{d\tilde{t}} = x(1-x)(x-\beta) - xy,$$

$$\frac{dy}{d\tilde{t}} = \hat{\eta}xy - \frac{\hat{h}y}{\omega+y} - \hat{\delta}y,$$
(2)

where $\beta = m/K$, $\hat{\eta} = b/r$, $\hat{h} = acE/r^2k_2K^2$, $\omega = ak_1E/rk_2K$, and $\hat{\delta} = d/rK$. We note the number of parameters are reduced from ten to five, which means model (2) is simpler than model (1).

In the current development of biological modeling, the first-order model is considered insufficient in describing a more realistic circumstance since the next state of the biological condition depends only on the current state [13, 14]. Therefore, we need a better mathematical operator to include all previous biological conditions in expressing the current condition which is known as the memory effects. The precise operator for this purpose is given by the fractional-order derivative [6, 13, 15, 16]. The most used operator is Caputo fractional-order derivative ${}^{C}\mathscr{D}_{t}^{\alpha}$ defined by

$${}^{C}\mathscr{D}_{t}^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau,$$

where $\alpha \in (0,1]$ is the order of the derivative and $\Gamma(\cdot)$ is Euler Gamma function [17]. Now, the first-order derivatives at the left-hand side of model (2) are replaced by the Caputo fractional-order derivative. The following model is achieved.

$${}^{C}\mathscr{D}_{t}^{\alpha}x = x(1-x)(x-\beta) - xy,$$

$${}^{C}\mathscr{D}_{t}^{\alpha}y = \hat{\eta}xy - \frac{\hat{h}y}{\omega+y} - \hat{\delta}y.$$
(3)

Since the derivatives at the left-hand side of model (3) have dimensions time^{$-\alpha$} while some parameters at the righthand side of model (3) have dimensions time⁻¹, we have inconsistency in using time unit here. Hence, we modify the parameters in model (3) to rectify this increasistency as follows.

$$\overline{}^{C}\mathscr{D}_{t}^{\alpha}x = x(1-x)(x-\beta) - xy,$$

$$C^{C}\mathscr{D}_{t}^{\alpha}y = \hat{\eta}^{-\alpha}xy - \frac{\hat{h}^{-\alpha}y}{\omega+y} - \hat{\delta}^{-\alpha}y.$$
(4)

Now, to make more simpler parameters form, we take $\eta = \hat{\eta}^{-\alpha}$, $h = \hat{h}^{-\alpha}$, and $\delta = \hat{\delta}^{-\alpha}$. Thus, the model becomes

$${}^{C}\mathscr{D}_{t}^{\alpha}x = x(1-x)(x-\beta) - xy \equiv F_{1}(x,y),$$

$${}^{C}\mathscr{D}_{t}^{\alpha}y = \eta xy - \frac{hy}{\omega+y} - \delta y \equiv F_{2}(x,y).$$
(5)

By considering all of the adv 34 ges possessed by the fractional-order operator as described above, we prefer to study the dynamical behaviors of the model (5) including the local asymptotic stability, the occurrence of transcritical and Hopf bifurcations, and some numerical simulations which support the analytical results such as the bifurcation diagram driven by the Allee threshold, the related phase portraits, and their time series.



In this section, we investigate the dynamical behaviors of model (5) involving the existence of equilibrium points, their locity tability, and the occurrence of bifurcations. We first identify the equilibrium point and their existence by solving $F_1(x, y) = F_2(x, y) = 0$. Thus, we have

$$x = 0$$
, or (6)

$$(1-x)(x-\beta) - y = 0$$
, and; (7)

$$y = 0, \text{ or}$$
 (8)

$$\eta x - \frac{h}{\omega + y} - \delta = 0. \tag{9}$$

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From eqs. (6) and (8), we obtain the origin point $E_0 = (0,0)$ which describes the extinction of both prey and predator. Moreover, by solving eqs. (6) and (9) we achieve $y = -(h + \delta \omega)/\delta < 0$ which contradicts with the biological condition. This means, there is no equilibrium point here. Now, we find out the existence of equilibrium point given by eqs. (7) and (8). By substituting eq. (8) to eq. (7), we achieve $(1-x)(x-\beta) = 0$ and hence we have x = 1 or $x = \beta$. Therefore, we acquire two axial equilibrium points $E_1 = (1,0)$ and $E_2 = (\beta,0)$. These equilibrium points express the predo or extinction point. Denotes these three equilibrium points E_i , i = 1, 2, 3 are always exist. Finally, we investigate the existence of equilibrium point given by eqs. (7) and (9). From eq. (7), we have

$$y = (1 - x)(x - \beta).$$
 (10)

By substituting eq. (10) to (9), we obtain a cubic equation

$$x^3 + 3\xi_1 x^2 + 3\xi_2 x + \xi_3 = 0 \tag{11}$$

where

$$\xi_1 = -\frac{\eta + \beta \eta + \delta}{3\eta}, \ \xi_2 = \frac{(\beta \eta + \beta \delta + \delta) - \eta \omega}{3\eta}, \ \text{and} \ \xi_3 = \frac{h + \delta \omega - \beta \delta}{\eta}.$$

Thus, we acquire the co-existence point given by $E_3 = (\hat{x}, \hat{y})$ where $\hat{y} = (1 - \hat{x})(\hat{x} - \beta)$ and \hat{x} is a positive solution of eq. (11). The interior point represents the existence of both prey and predator. To identify its existence condition, from eq. (10), the interior point E_3 exists if $\hat{x} > 0$ and; $\beta < \hat{x} < 1$ or $1 < \hat{x} < \beta$. Now, by the transformation $z = x + \xi_1$, we have

$$z^3 + 3pz + q = 0, (12)$$

where $p = \xi_2 - \xi_1^2$, and $q = \xi_3 - 3\xi_1\xi_2 + 2\xi_1^3$. Following the Cardano's method as in [18], we have the following lemma.

- Lemma 1. Equation (12) has (i) a positive root if: (a) q < 0 (single), or (b) q = 0 and p < 0 (unique), or (c) q > 0, p < 0, and $q^2 + 4p^3 = 0$ (multiplicity two).

(*ii*) two positive root if q > 0, p < 0, and $q^2 + 4p^3 < 0$. Furthermore, the local stability of those equilibrium points are investigated. By applying linearization to model (5), we have the Jacobian matrix as follows.

$$J(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} -3x^2 + 2(1+\beta)x - y - \beta & -x\\ \eta \mathbf{y} & \eta \mathbf{x} - \frac{\omega h}{(\omega+\mathbf{y})^2} - \delta \end{bmatrix}.$$
 (13)

The local stability for each equilibrium point utilizing the Jacobian matrix (13) given by the following theorems.

Theorem 1. The origin point $E_0 = (0,0)$ is always locally asymptotically stable.

Proof. By evaluating Jacobian matrix (13) at E_0 , we obtain

$$J(x,y)|_{E_0} = \begin{bmatrix} -\beta & 0\\ 0 & -\frac{h+\delta\omega}{\omega} \end{bmatrix},$$

which gives eigenvalues $\lambda_1 = -\beta$ and $\lambda_2 = -\frac{h+\delta\omega}{\omega}$. Since all parameter are positive, we have $\lambda_i < 0$, i = 1, 2, which means $|\arg(\lambda_i)| > \alpha \pi/2$ i = 1, 2. Therefore, E_0 is always locally asymptotically stable.

Theorem 2. Let $\hat{\beta} = \frac{h + \delta \omega}{\eta \omega}$. The local dynamics of the axial equilibrium points are given as follows. (i) If max $\{1, \beta\} < \hat{\beta}$ and

(i.i) $\beta < 1$ then E_1 is locally asymptotically stable and E_2 is a saddle point.

(*i.ii*) $\beta = 1$ then $E_1 \underset{3}{\underset{3}{}} E_2 = (1,0)$ is non-hyperbolic.

(i.iii) $\beta > 1$ then E_2 is locally asymptotically stable and E_1 is a saddle point.

- (ii) If $\hat{\beta} = 1$ or $\hat{\beta} = \beta$ then E_1 or E_2 is non-hyperbolic respectively.
- (iii) If $\min\{1,\beta\} > \hat{\beta}$ then both axial equilibrium points are unstable.

Proof. When $E_1 = (1,0)$ and $E_2 = (\beta,0)$, the Jacobian matrix (13) becomes

$$J(\mathbf{x},\mathbf{y})|_{E_1} = \begin{bmatrix} \beta - 1 & -1 \\ 0 & \left(1 - \hat{\beta}\right) \eta \end{bmatrix},$$
(14)

and

$$J(x,y)|_{E_2} = \begin{bmatrix} -(\beta-1)\beta & -\beta \\ 0 & (\beta-\hat{\beta})\eta \end{bmatrix},$$
(15)

where each Jacobian matrices give eigenvalues: $\lambda_{1a} = \beta - 1$ and $\lambda_{2a} = (1 - \hat{\beta})\eta$; $\lambda_{1b} = -(\beta - 1)\beta$ and $\lambda_{2b} = (\beta - \hat{\beta})\eta$, respectively.

- (i) When $\max\{1,\beta\} < \hat{\beta}$, we confirm that the second eigenvalues of both (14) and (15) are negative and hence $|\arg(\lambda_{2a,2b})| = \pi > \alpha \pi/2$. When $\beta < 1$, we get $\lambda_{1a} < 0$ and $\lambda_{1b} > 0$. Thus, $|\arg(\lambda_{1a})| = \pi > \alpha \pi/2$ and $|\arg(\lambda_{1b})| = 0 < \alpha \pi/2$. Hence, E_1 is locally asymptotically stable and E_2 is a saddle point. Furthermore, when $\beta = 1, E_1$ and E_2 merge to a single predator extinction point $E_{1,2} = (1,0)$ which gives $\lambda_{1a,1b} = 0$ and $|\arg(\lambda_{1a,1b})| = \alpha \pi/2$. Thus, $E_{1,2}$ is a non-hyperbolic point. At last, when $\beta > 1$, we easily confirm that the stability of E_1 and E_2 have the opposite sign with $\beta < 1$.
- (ii) For $\hat{\beta} = 1$ and $\hat{\beta} = \beta$, we have $|\arg(\lambda_{2a})| = \alpha \pi/2$ and $|\arg(\lambda_{2b})| = \alpha \pi/2$, respectively. Thus, E_1 or E_2 becomes non-hyperbolic points.
- (iii) When min $\{1,\beta\} > \hat{\beta}$, we achieve $\lambda_{2a} > 0$ and λ_{2b} . Hence, $|\arg(\lambda_i)| = 0 < \alpha \pi/2$, i = 2a, 2b. Therefore, both E_1 and E_2 are unstable point.

 \Box

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Theorem 3. Suppose that one of the following statements hold.

(*i*) $\varphi_1 < 0$ and $\varphi_2 > 0$, or;

(*ii*)
$$\varphi_1^2 < 4\varphi_2$$
, $\varphi_1 > 0$, and $\alpha < \hat{\alpha}$,

where

$$\varphi_{1} = ((1+\beta) - 2\hat{x})\hat{x} + \frac{h\hat{y}}{(\omega+\hat{y})^{2}},$$
$$\varphi_{2} = \left(\frac{((1+\beta) - 2\hat{x})h}{(\omega+\hat{y})^{2}} + \eta\right)\hat{x}\hat{y},$$
$$\hat{\alpha} = \frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{4\varphi_{2} - \varphi^{2}}}{\varphi_{1}}\right),$$

then the co-existence point $E_3 = (\hat{x}, \hat{y})$ is locally asymptotically stable.

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Proof. By evaluating the Jacobian matrix (13) at E_3 , we achieve

$$J(x,y)|_{E_3} = \begin{bmatrix} ((1+\beta)-2\hat{x})\hat{x} & -\hat{x} \\ & & & \\ \eta\hat{y} & & \\ & & & \\ \eta\hat{y} & & \\ & & & \\ \hline & & & \\ & & & \\ & & & \\ \end{pmatrix}_{E_3} = \begin{bmatrix} ((1+\beta)-2\hat{x})\hat{x} & & & \\ & & &$$

which gives characteristic polynomial equation $\lambda^2 - \varphi_1 \lambda + \varphi_2 = 0$ and eigenvalues

$$\lambda_1 = rac{arphi_1}{2} - rac{\sqrt{arphi_1^2 - 4arphi_2}}{2}, \ \lambda_2 = rac{arphi_1}{2} + rac{\sqrt{arphi_1^2 - 4arphi_2}}{2}.$$

Therefore, $\lambda_{1,2}$ could be real or complex numbers depend on the sign of $\varphi_1^2 - \varphi_2$. If $\varphi_1 < \bigcirc_3$ nd $\varphi_2 > 0$ then $|\arg(\lambda_i)| > \frac{\alpha \pi}{2}$, i = 1, 2 for every sign of $\varphi_1^2 - \varphi_2$. Furthermore, if $\varphi_1^2 < 4\varphi_2$ and $\varphi_1 > 0$, we have a pair of complex conjugate eigenvalues $\lambda_{1,2}$ with positive real parts. Hence, since $\alpha < \hat{\alpha}$, we obtain $|\arg(\lambda_i)| > \frac{\alpha \pi}{2}$, i = 1, 2. Thus, Theorem 3 is completely proven.

The interesting dynamics are shown by Theorems 2 and 3. In Theorem 2(i), there exists a condition such that the local stabilities of E_1 and E_2 have the opposite sign and exchange their stability when the Allee threshold β is varied. Furthermore, from Theorem 3, we also have a condition that the dynamical behavior around E_3 changes when the order (α) is varied. These circumstances lead to the occurrence of two bifurcations namely transcritical and Hopf bifurcations. We give the following theorems to show their existence.

Theorem 4. Let $\beta^* = 1$ and max $\{1, \beta\} < \hat{\beta}$. E_1 and E_2 switched their stability via transcritical bifurcation when β crosses β^* .

Proof. Denote when $\frac{h+\delta\omega}{\eta\omega} > \max\{1,\beta\}$, the Jacobian matrix for both E_1 and E_2 have a negative eigenvalue such as 29 $|\arg(\lambda_2)| > \frac{\alpha\pi}{2}$ for each Jacobian matrix (14) and (15). Therefore, the stability of both equilibrium points depend on the others eigenvalues. According to Theorem 2, when $\beta < \beta^*$, E_1 is locally asymptotically stable while E_2 becomes a saddle point. For $\beta = \beta^*$, E_2 merges with E_1 and the equilibrium point becomes non-hyperbolic. Finally, when $\beta > \beta^*$, E_1 becomes a saddle point while E_2 becomes locally asymptotically stable. This ends the proof.

Theorem 5. Suppose that $\varphi_1^2 < 4\varphi_2$ and $\varphi_1 > 0$. The interior point $E_3 = (\hat{x}, \hat{y})$ changes its stability via Hopf bifurcation when α passes through $\hat{\alpha}$.

Proof. According to Theorem 3, since $\varphi_1^2 < 4\varphi_2$ and $\varphi_1 > 0$, the Jacobian matrix (13) evaluated at E_3 has a pair of complex conjugate eigenvalues with positive real parts. Furthermore, we can easily show that

$$\begin{split} m(\hat{\alpha}) &= \frac{\hat{\alpha}\pi}{2} - \min_{1 \le i \le 2} |\arg(\lambda_i)| \\ &= \frac{2}{\pi} \tan^{-1} \left(\frac{\sqrt{4\varphi_2 - \varphi^2}}{\varphi_1} \right) - \frac{2}{\pi} \tan^{-1} \left(\frac{\sqrt{4\varphi_2 - \varphi^2}}{\varphi_1} \right) = 0, \text{ and} \\ \frac{dm(\alpha)}{d\alpha} \bigg|_{\alpha = \hat{\alpha}} &= \frac{\pi}{2} \ne 0. \end{split}$$

Therefore, all conditions given by Theorem 4 in [13] are satisfied. Hence, E_3 losses its stability via Hopf bifurcation, where α is the bifurcation parameter and α^* is the bifurcation point.

NUMERICAL SIMULATIONS

Now, we will show numerically the dynamical behaviors of model (5) based on the analytical results including the bifurcation diagram and the local stability. To obtain the numerical solution, the predictor-corrector scheme for



 $h = 0.1, \omega = 0.1, \text{ and } \alpha = 0.9.$

fract al-order differential equation proposed by Diethelm is applied [19]. We divide the simulations into two parts in impact of the Allee effect and the order of the derivative to the diparticle behaviors around the predator extinction point and the co-existence point, respectively. First, to show the impact of the Allee effect, we fix some parameters as follows.

$$\eta = 0.5, \ \delta = 0.1, \ h = 0.1, \ \text{and}, \ \omega = 0.1.$$
 (16)

We ensure that the co-existence point does not exist. From Theorems 1 and 2, we also confirm that the stability of E_i , i = 0, 1, 2 does not depend on α . We choose $\alpha = 0.9$ as the order of the derivative. Furthermore, to β estigate the changes in dynamical behavior of model (5) driven by the Allee threshold, we vary β in interval [0,3]. We obtain the bifurcation diagram as in Figure 1(a). There are two points which cause the change in behavior of model (5) namely $\beta^* = 1$ and $\hat{\beta} = 2.2$. When $0 \le \beta < \beta^*$, E_1 is asymptotically stable while E_2 is unstable. After β crosses β^* , E_1 and E_2 exchange their stability via trascritical bifurcation. These circumstances confirm Theorems 2 and 4. The local dynamics around E_1 and E_2 are maintained in interval $\beta^* < \beta < \hat{\beta}$. Next, when β crosses $\hat{\beta}$, both 44 and E_2 become unstable. We present the phase portrait of those conditions in Figure 1(b) to 1(d). In addition, in Figure 1(b) and 1(c), model (5) undergoes the bistability conditions which describe the sensitivity of solution based on their initial



Figure 2. The bifurcation diagram driven by the order of the derivative (α) using parameter values $\beta = 0.4, \eta = 0.5, \delta = 0.1, h = 0.1, \text{ and } \omega = 0.3.$

conditions. We have a pair of fairly close initial conditions that convergent to the distinct equilibrium points. This means, if the ratio between prey and predator does not large enough, both populations will egonet.

Finally, we will show the influence of the order- α to the stability of co-existence point. The following parameter values are setted.

$$\beta = 0.4, \ \eta = 0.5, \ \delta = 0.1, \ h = 0.1, \ \text{and} \ \omega = 0.1.$$
 (17)

Therefore, all types of equilibrium points exist. According to Theorems 1 to 3, We have an always locally asymptotically stable E_0 , a pair of unstable predator extinction points E_1 and E_2 , and a conditionally stable co-existence point E_3 which depends on the value of α . Since the stability of E_3 is determined by the order- α , we portray the bifurcation diagram of model (5) driven by α in interval [0.8, 1]. See Figure 2. For $0.8 < \alpha < \alpha^*$, $\alpha^* \approx 0.902$, E_3 is locally asymptotically stable. When α crosses α^* , E_3 losses its stability via Hopf bifurcation and nearby solution convergent to a periodic signal. The Figure 3 is given to provide the corresponding phase portrait and time series. Apart from Hopf bifurcation, we also have two types of bistability conditions here which are shown by Figure 3(a) to 3(d). When $\alpha = 0.89$, the bistability condition exists as an impact of two locally asymptotically stable equilibrium points i.e. E_0 and E_3 . The bistability condition is still maintained when $\alpha = 0.98$ but involving a locally asymptotically stable E_0 and a stable limit-cycle that occurs via Hopf bifurcation. This means for the initial condition nearby the co-existence point, the existence of both populations are maintained in two ways ie. (i) convergent to a constant value (ii) eventually change periodically, which depend on how strong the effect of memory affects their growth rate. Nevertheless, as a result of the Allee effect, both populations may be extinct for some initial conditions although the co-existence point or the limit-cycle is stable.

CONCLUSION

We have completely investigated the dynamical behaviors of a predator-prey model with strong Allee effect in prey and Michaelis-Menten type of predator harvesting with Caputo fractional order derivative. Four equilibrium points are obtained namely an origin point, a pair of the predator extinction points, and the co-existence point. The origin point is always locally asymptotically stable while others are condige ally asymptotically stable. Some interesting phenomena also have been found in addition to their local stability such as the existence of transcritical bifurcation, Hopf bifurcation, and bistability condition. We strengthen the analytical results by providing some numerical simulations. We also have shown numerically that although the co-existence point is unstable, the existence of both populations is still maintained periodically.



Figure 3. The phase portraits and time series using parameter values $\beta = 0.4$, $\eta = 0.5$, $\delta = 0.1$, h = 0.1, and $\omega = 0.3$.

In this paper, some important results are not investigated such as the global dynamics, the sensitivity of α , and the difference between the fractional-order model with the integer ones. Thus, investigating these dynamical behaviors will be interesting for the next research.



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