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Original Research

The Role of Alternative Food Quality in Predator Persistence: Stability and Forward Bifurcation Analysis of a Predator–Prey Model

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Abstract

This study develops a two-dimensional predator–prey model incorporating logistic prey growth, prey group defense, alternative food, and ecological costs to the predator. The novelty of the model lies in the introduction of an alternative food quality parameter, which distinguishes the availability of alternative food from its biological effectiveness in supporting predator growth. The analysis focuses on the existence of equilibrium points, local stability, and forward bifurcation. The analytical results show that the model admits a trivial equilibrium, a predator-free equilibrium, and an interior equilibrium under certain existence conditions. The trivial equilibrium is always unstable, whereas the predator-free equilibrium is locally asymptotically stable when the quality of alternative food is below a critical value and loses its stability when the quality parameter exceeds this threshold. At the critical value, the system undergoes a forward bifurcation characterized by a stability transition from the predator-free equilibrium to a positive interior equilibrium. Numerical simulations confirm the analytical findings and illustrate three main scenarios: predator extinction when the alternative food quality is low, stable coexistence when the alternative food quality is high, and a stability transition through a bifurcation diagram as the alternative food quality parameter crosses its critical value. These results indicate that alternative food quality acts as a threshold parameter determining predator persistence and the transition from predator extinction to stable coexistence.

Keywords: Alternative food quality; Forward bifurcation; Local stability; Predator–prey model

1. Introduction

Mathematical models in population dynamics have become essential tools for understanding ecological interactions between two or more species. Early studies on population growth can be traced back to the ideas of

Malthus, who described population growth as exponential under conditions of unlimited resources [1]. Subsequently, Lotka and Volterra independently developed the classical predator–prey formulation, which serves as a fundamental basis for modeling interactions between prey and predators [2, 3]. Although the Lotka–Volterra model has strong historical and theoretical significance, it still relies on several simplifying assumptions, such as unbounded prey growth and linear predation response. Therefore, various extended models have been developed to incorporate more realistic ecological mechanisms, including logistic growth, predator functional responses, and limitations on population growth. The Leslie–Gower [4] and Rosenzweig–MacArthur [5] models are two important examples that extend the classical framework by considering interaction structures that better reflect ecological phenomena.

In subsequent developments, predator–prey models have been extended to investigate various biological mechanisms that influence system stability. Several studies have incorporated disease infection and fear effects in prey populations, predator harvesting, and eco-epidemiological structures [6–10]. Other studies have considered memory effects, fractional-order dynamics, predator age structure, intraspecific competition, supplementary food, and nonlinear harvesting to capture more complex population dynamics [10–14]. In addition, mechanisms such as poisoning of predators [15], anti-predator behavior [16, 17], prey protection [18], fear effects [19], Allee effects [20], and nonlinear functional responses have also been used to explain stability changes [21–24] and the emergence of various dynamic phenomena, such as bifurcation, oscillations, and limit cycles [25]. These developments indicate that the stability of predator–prey systems is strongly influenced by the biological mechanisms incorporated into the model.

One ecological mechanism that has attracted increasing attention in recent years is the presence of alternative food or supplementary food for predators. In natural ecosystems, predators do not always depend entirely on their primary prey. When the primary prey decreases, becomes difficult to obtain, or develops strong protective mechanisms, predators may exploit other food sources to maintain their populations. The availability of alternative food can alter predation pressure on the primary prey, increase the survival prospects of predators, and affect the stability of the system's equilibrium points. Several studies have shown that supplementary food can generate complex dynamics, including stability changes, memory effects, bistability, chaos, and shifts in interaction patterns in predator–prey models [25–28]. However, most models involving alternative food still emphasize the aspect of food availability or quantity, while the quality of alternative food has not been explicitly separated from the amount of food available.

Biologically, however, alternative food has not only a quantitative dimension but also a qualitative one. Two environments may provide the same amount of alternative food, but its nutritional value, accessibility, toxic risk, or contribution to predator reproduction may differ. In other words, abundant alternative food does not necessarily provide a substantial biological contribution to predator growth. Conversely, alternative food that is limited in quantity but high in quality may have a significant effect on predator persistence. Therefore, distinguishing between the availability and quality of alternative food is important in constructing a more realistic predator–prey model.

A recent predator–prey model developed by Resmawan and Achmad [29] combines three important mechanisms: logistic prey growth, prey group defense, and the presence of alternative food accompanied by ecological costs to the predator. Prey group defense is represented by a nonlinear term in the denominator of the functional response, while the ecological cost of consuming alternative food is modeled as an increase in predator mortality. The model makes an important contribution by showing that alternative food should not only be viewed as an additional resource that benefits predators, but also as a factor that may impose ecological costs detrimental to predators. Nevertheless, the model still assumes that the contribution of alternative food to predator growth is entirely determined by its availability, without distinguishing the quality or biological effectiveness of the alternative food.

Based on this gap, the present study proposes the development of a predator–prey model by introducing an alternative food quality parameter for predators. This ecological factor is used to distinguish between the amount of alternative food available and the effectiveness of that food in supporting predator reproduction. Thus, the contribution of alternative food to predator growth is no longer represented solely by the food availability component, but is modified by incorporating a quality factor for the available supplementary food. This formulation provides a simple extension of the model proposed by Resmawan and Achmad [29], while retaining strong biological relevance because it represents the distinction between the availability and quality of food resources.

The main objective of this study is to analyze the influence of alternative food quality on the dynamics of the predator–prey system. The analysis is carried out by determining the existence of equilibrium points, local stability, and bifurcation behavior that describes the transition from a predator-free state to stable coexistence between the two populations. Numerical simulations are presented to confirm the analytical results and provide biological interpretations of the role of alternative food quality in supporting predator persistence.

The main contributions of this study can be summarized in three aspects. First, this study introduces an alternative food quality parameter into a predator–prey model with prey group defense and predator ecological costs. Second, this study shows that alternative food quality can serve as a key parameter determining the stability of the predator-free equilibrium and the emergence of an interior equilibrium. Third, through bifurcation analysis and numerical simulations, this study demonstrates that increasing the quality of alternative food can shift the system dynamics from predator extinction to stable coexistence. Thus, this study provides a simple but meaningful extension of previous predator–prey models, particularly by explaining that predator persistence is determined not only by the availability of alternative food but also by its ecological quality.

2. Model

The model in this study refers to the predator–prey model with prey group defense, alternative food, and predator ecological costs as given in [29]. Let $N(t)$ denote the prey population density and $P(t)$ denote the predator population density at time t. The reference model is given by

$$\begin{aligned} \frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right)-\frac{\alpha NP}{\beta+\kappa mA+\mu N^2}, \\ \frac{dP}{dt} &= \frac{\zeta(N+mA)P}{\beta+\kappa mA+\mu N^2}-(\delta+\theta mA)P. \end{aligned} \tag{1}$$

Model (1) incorporates logistic prey growth, a nonlinear predation response, the effect of alternative food, prey group defense, and predator ecological costs. The component μN^2 in the denominator indicates that higher prey density strengthens group defense, thereby reducing predation effectiveness. Meanwhile, alternative food A plays two roles: it affects the predation process through κmA and increases the effective mortality of predators through $\delta + \theta mA$. In this model, alternative food also contributes to predator growth through the component mA.

In model (1), alternative food is assumed to contribute fully to predator growth solely based on its availability. Biologically, however, alternative food is determined not only by its quantity but also by its quality. Two environments may provide the same amount of alternative food, but their nutritional value, accessibility, or effectiveness in supporting predator reproduction may differ. Therefore, this study introduces a new parameter q, with $0 < q \leq 1$, which represents the quality of alternative food. The modification of the contribution of alternative food to predator growth is represented by replacing mA with qmA. If $q = 1$, the alternative food is assumed to have full quality, and the model reduces to the assumption of the reference model. If $0 < q < 1$, the alternative food remains available, but its effective contribution to predator growth becomes lower. Thus, the new model can distinguish between the availability of alternative food A and its biological quality q. Based on this development, the new model studied in this research is

$$\begin{aligned} \frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right)-\frac{\alpha NP}{\beta+\kappa mA+\mu N^2}, \\ \frac{dP}{dt} &= \frac{\zeta(N+qmA)P}{\beta+\kappa mA+\mu N^2}-(\delta+\theta mA)P. \end{aligned} \tag{2}$$

Model (2) provides a more realistic interpretation of alternative food, in which alternative food may influence the predation process and predator ecological costs, while its benefit to predator reproduction depends on its quality. Therefore, the parameter q becomes a key parameter that may determine whether predators can persist or go extinct.

To simplify the analysis, define

$$B = \beta + \kappa mA \qquad \text{and} \qquad M = \delta + \theta mA.$$

Since A is assumed to be constant, B and M are positive constants. Thus, model (2) can be written as

$$\begin{aligned} \frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right)-\frac{\alpha NP}{B+\mu N^2}, \\ \frac{dP}{dt} &= P\left[\frac{\zeta(N+qmA)}{B+\mu N^2}-M\right]. \end{aligned} \tag{3}$$

All analyses in the following sections are carried out based on model (3), with

$$r, K, \alpha, \beta, \kappa, m, A, \mu, \zeta, \delta, \theta > 0 \qquad \text{and} \qquad 0 < q \leq 1.$$

The descriptions of the model parameters are given in Table 1.

Table 1: Model parameters and descriptions.

Parameter	Description
$N(t)$	Prey population density at time t
$P(t)$	Predator population density at time t
r	Intrinsic growth rate of prey
K	Environmental carrying capacity of prey
α	Predation rate coefficient
β	Basic predation saturation constant
κ	Factor representing the effect of alternative food on predation saturation
m	Predator efficiency in finding alternative food
A	Availability of alternative food
μ	Strength of prey group defense
ζ	Biomass conversion efficiency into predator reproduction
δ	Natural mortality rate of predators
θ	Ecological cost of utilizing alternative food
q	Quality of alternative food

3. Results and Discussions

This section discusses the basic properties, existence of equilibrium points, local stability, and forward bifurcation of model (3). Before analyzing the equilibrium points, it is first shown that model (3) is well-defined both mathematically and biologically. Mathematically, the system must possess an existing and unique solution. Biologically, since $N(t)$ and $P(t)$ represent the prey and predator population densities, respectively, any solution starting from nonnegative initial conditions must remain nonnegative for all time during which the solution is defined.

3.1 Fundamental Properties of the Model

Two preliminary aspects required before the equilibrium analysis are the well-posedness of the solution and the consistency of the solution with the biological meaning of the model variables. These aspects are stated in Propositions 1 and 2.

Proposition 1 (Existence and Uniqueness of Solutions)**.** *For every initial condition*

$$(N(0), P(0)) = (N_0, P_0) \in \mathbb{R}^2_{\geq 0},$$

model equation (3) has a locally existing and unique solution on a time interval $[0, T_{\max})$.

Proof. Let the right-hand sides of model equation (3) be denoted by $F(N, P)$ and $G(N, P)$. Since

$$B = \beta + \kappa mA > 0 \qquad \text{and} \qquad \mu > 0,$$

it follows that

$$B + \mu N^2 > 0$$

for every $N \in \mathbb{R}$. Hence, the right-hand side of model equation (3) has no singularity on $\mathbb{R}^2$. The functions $F(N,P)$ and $G(N,P)$ are rational functions with strictly positive denominators, so they are continuous and have continuous partial derivatives with respect to N and P on $\mathbb{R}^2$. Consequently, the vector field

$$H(N,P) = \big(F(N,P), G(N,P)\big)$$

is continuous and locally Lipschitz on $\mathbb{R}^2$. By the existence and uniqueness theorem for ordinary differential equations, for every initial condition

$$(N_0, P_0) \in \mathbb{R}^2_{\geq 0},$$

there exists a local solution $(N(t), P(t))$ that is unique on a time interval $[0, T_{\max})$. Therefore, model equation (3) has a locally existing and unique solution. □

Proposition 2 (Positivity of Solutions)**.** *If the initial conditions satisfy*

$$N(0) = N_0 \geq 0, \qquad P(0) = P_0 \geq 0,$$

then the solution of model (3) *satisfies*

$$N(t) \geq 0, \qquad P(t) \geq 0,$$

for every $t \geq 0$ as long as the solution is defined. In other words, the nonnegative quadrant

$$\mathbb{R}^2_{\geq 0} = \{(N,P) : N \geq 0,\ P \geq 0\}$$

is a positively invariant region for model (3)*.*

Proof. From the structure of model (3), the first equation contains the factor N, while the second equation contains the factor P. Therefore, on the boundary $N = 0$, we obtain

$$\left.\frac{dN}{dt}\right|_{N=0} = 0,$$

and on the boundary $P = 0$, we obtain

$$\left.\frac{dP}{dt}\right|_{P=0} = 0.$$

This shows that the vector field on the boundary of the nonnegative quadrant does not point outward from $\mathbb{R}^2_{\geq 0}$. In other words, a solution starting on the axis $N = 0$ cannot enter the region $N < 0$, and a solution starting on the axis $P = 0$ cannot enter the region $P < 0$. Furthermore, model (3) can be written in the form

$$\frac{dN}{dt} = N\Phi_1(N,P),$$

and

$$\frac{dP}{dt} = P\Phi_2(N,P),$$

where Φ_1 and Φ_2 are continuous on $\mathbb{R}^2_{\geq 0}$. Hence, for $N_0 > 0$ and $P_0 > 0$, we obtain

$$N(t) = N_0 \exp\left(\int_0^t \Phi_1(N(s), P(s))\, ds\right) > 0,$$

and

$$P(t) = P_0 \exp\left(\int_0^t \Phi_2(N(s), P(s))\, ds\right) > 0.$$

Meanwhile, if $N_0 = 0$, then $N(t) = 0$, and if $P_0 = 0$, then $P(t) = 0$, as long as the solution is defined. Thus, every solution starting from nonnegative initial conditions remains in the nonnegative quadrant for all $t \geq 0$. Therefore, $\mathbb{R}^2_{\geq 0}$ is a positively invariant region for model (3). □

3.2 Equilibrium Points and Their Existence

The equilibrium points of model (3) are obtained by simultaneously solving the following system of equations:

$$\begin{aligned} N\left[r\left(1-\frac{N}{K}\right)-\frac{\alpha P}{B+\mu N^2}\right]&=0,\\ P\left[\frac{\zeta(N+qmA)}{B+\mu N^2}-M\right]&=0. \end{aligned} \tag{4}$$

From system (4), three types of equilibrium points are obtained, namely the trivial equilibrium point, the predator-free equilibrium point, and the interior equilibrium point. The existence of these equilibrium points is presented in Theorem 1.

Theorem 1. *Let* $N_{2,3}=\dfrac{\zeta\pm\sqrt{\Delta}}{2M\mu}$, $N_4=\dfrac{\zeta}{2M\mu}$, *where* $\Delta=\zeta^2-4M\mu\left(MB-\zeta qmA\right)$. *Furthermore, define*

$$P_i=\frac{r\left(B+\mu N_i^2\right)\left(K-N_i\right)}{K\alpha},\qquad i=2,3,4.$$

Model (3) *has three types of equilibrium points as follows:*

1. *The trivial equilibrium point* $E_0=(0,0)$, *which indicates the absence of prey and predator populations in the ecosystem. This point always exists in* $\mathbb{R}^2_{\geq 0}$.
2. *The predator-free equilibrium point* $E_1=(K,0)$, *which represents the condition in which the predator population becomes extinct while the prey population remains at the environmental carrying capacity. This point always exists in* $\mathbb{R}^2_{\geq 0}$.
3. *The interior equilibrium points* $E_i=(N_i,P_i),\quad i=2,3,4$, *which indicate the coexistence of prey and predator populations. The interior equilibrium points exist if and only if* $K>N_i$ *and* $\Delta>0$, *which is equivalent to*

$$q>\frac{4M^2\mu B-\zeta^2}{4M\mu\zeta mA}.$$

If $\Delta=0$, *which is equivalent to*

$$q=\frac{4M^2\mu B-\zeta^2}{4M\mu\zeta mA},$$

then there exists a unique interior equilibrium point in the form of a double root, namely

$$E_4=(N_4,P_4),\qquad N_4=\frac{\zeta}{2M\mu}.$$

Proof. From the second equation of system (4), we obtain $P=0$ or

$$\frac{\zeta(N+qmA)}{B+\mu N^2}-M=0.$$

Case 1: $P=0$.

From the first equation of system (4), we obtain

$$N\left[r\left(1-\frac{N}{K}\right)\right]=0.$$

Since $r>0$, it follows that

$$N=0\qquad\text{or}\qquad N=K.$$

Thus, two boundary equilibrium points are obtained, namely

$$E_0=(0,0)\qquad\text{and}\qquad E_1=(K,0).$$

Both equilibrium points always exist in $\mathbb{R}^2_{\geq 0}$.

Case 2: $P > 0$.

In this case, the second equation of system (4) requires

$$\frac{\zeta(N_i + qmA)}{B + \mu N_i^2} = M.$$

This equation is equivalent to the quadratic equation

$$M\mu N_i^2 - \zeta N_i + MB - \zeta qmA = 0.$$

The solutions of this quadratic equation are

$$N_{2,3} = \frac{\zeta \pm \sqrt{\Delta}}{2M\mu},$$

where

$$\Delta = \zeta^2 - 4M\mu\left(MB - \zeta qmA\right).$$

If $\Delta > 0$, then there are two distinct real roots, namely N_2 and N_3. If $\Delta = 0$, then there is a double real root given by

$$N_4 = \frac{\zeta}{2M\mu}.$$

Furthermore, for $N_i > 0$, the first equation of system (4) gives

$$r\left(1 - \frac{N_i}{K}\right) - \frac{\alpha P_i}{B + \mu N_i^2} = 0.$$

Consequently,

$$P_i = \frac{r\left(B + \mu N_i^2\right)(K - N_i)}{K\alpha}, \qquad i = 2, 3.$$

Since

$$r > 0, \qquad B + \mu N_i^2 > 0, \qquad K > 0, \qquad \alpha > 0,$$

we have $P_i > 0$ if and only if $K > N_i$. Therefore, the biologically feasible interior equilibrium points are

$$E_i = (N_i, P_i), \qquad i = 2, 3, 4,$$

provided that $\Delta > 0$ and $K > N_i$. In this case, the condition $\Delta > 0$ is equivalent to

$$q > \frac{4M^2\mu B - \zeta^2}{4M\mu\zeta mA}.$$

For the case $\Delta = 0$, that is,

$$q = \frac{4M^2\mu B - \zeta^2}{4M\mu\zeta mA},$$

the unique interior equilibrium point is given by $E_4 = (N_4, P_4)$. □

Biologically, Theorem 1 shows that model (3) always has a trivial equilibrium point and a predator-free equilibrium point. The interior equilibrium emerges only when the quality of alternative food, represented by the parameter q, sufficiently supports predator growth, thereby allowing coexistence between the prey and predator populations.

3.3 Local Stability

The local stability analysis of the equilibrium points of model (3) is carried out using the Jacobian matrix. Let the right-hand sides of model (3) be denoted by $F(N,P)$ and $G(N,P)$. The Jacobian matrix of model (3) is given by

$$J(N,P) = \begin{pmatrix} F_N & F_P \\ G_N & G_P \end{pmatrix}.$$

Therefore, the Jacobian matrix for model (3) is

$$J(N,P) = \begin{pmatrix} r - \dfrac{2rN}{K} - \dfrac{\alpha P(B-\mu N^2)}{\left(B+\mu N^2\right)^2} & -\dfrac{\alpha N}{B+\mu N^2} \\ \dfrac{\zeta P\left(B-\mu N^2-2\mu qmAN\right)}{\left(B+\mu N^2\right)^2} & \dfrac{\zeta(N+qmA)}{B+\mu N^2} - M \end{pmatrix}. \tag{5}$$

The local stability of each equilibrium point is determined by the signs of the real parts of the eigenvalues of the Jacobian matrix (5) evaluated at the corresponding equilibrium point. The results of the local stability analysis are presented in the following theorems.

Theorem 2. *The trivial equilibrium point $E_0 = (0,0)$ is always unstable. More specifically, E_0 is a saddle point if*

$$\frac{\zeta qmA}{B} < M,$$

and it is nonhyperbolic if

$$\frac{\zeta qmA}{B} = M.$$

Proof. The Jacobian matrix at $E_0 = (0,0)$ is

$$J(E_0) = \begin{pmatrix} r & 0 \\ 0 & \dfrac{\zeta qmA}{B} - M \end{pmatrix},$$

which has the eigenvalues

$$\lambda_1 = r > 0$$

and

$$\lambda_2 = \frac{\zeta qmA}{B} - M.$$

Since $\lambda_1 > 0$, the equilibrium point E_0 is unstable. If

$$\frac{\zeta qmA}{B} < M,$$

then $\lambda_2 < 0$, so E_0 is a saddle point. If

$$\frac{\zeta qmA}{B} = M,$$

then $\lambda_2 = 0$, so E_0 is nonhyperbolic. □

Theorem 3. *The predator-free equilibrium point $E_1 = (K,0)$ is locally asymptotically stable if*

$$q < \frac{M\left(B+\mu K^2\right) - \zeta K}{\zeta mA},$$

is a saddle point if

$$q > \frac{M\left(B+\mu K^2\right) - \zeta K}{\zeta mA},$$

and is nonhyperbolic if

$$q = \frac{M\left(B+\mu K^2\right) - \zeta K}{\zeta mA}.$$

Proof. The Jacobian matrix at $E_1 = (K, 0)$ is

$$J(E_1) = \begin{pmatrix} -r & -\dfrac{\alpha K}{B + \mu K^2} \\ 0 & \dfrac{\zeta(K + qmA)}{B + \mu K^2} - M \end{pmatrix},$$

with the two eigenvalues given by

$$\lambda_1 = -r < 0$$

and

$$\lambda_2 = \frac{\zeta(K + qmA)}{B + \mu K^2} - M.$$

Since $\lambda_1 < 0$, the stability of E_1 is determined by the sign of λ_2. If

$$\frac{\zeta(K + qmA)}{B + \mu K^2} < M,$$

which is equivalent to

$$q < \frac{M\left(B + \mu K^2\right) - K\zeta}{mA\zeta},$$

then $\lambda_2 < 0$, so both eigenvalues are negative. Hence, E_1 is locally asymptotically stable. Conversely, if

$$q > \frac{M\left(B + \mu K^2\right) - \zeta K}{mA\zeta},$$

then $\lambda_2 > 0$. Since $\lambda_1 < 0$ and $\lambda_2 > 0$, E_1 is a saddle point. Furthermore, if

$$q = \frac{M\left(B + \mu K^2\right) - \zeta K}{mA\zeta},$$

then $\lambda_2 = 0$, so E_1 is nonhyperbolic. □

Theorem 4. *Let $E_i = (N_i, P_i), \quad i = 2, 3, 4$, be an interior equilibrium point of model (3), where $\Delta = \zeta^2 - 4M\mu\left(MB - \zeta qmA\right)$, $N_{2,3} = \dfrac{\zeta \pm \sqrt{\Delta}}{2M\mu}$, $N_4 = \dfrac{\zeta}{2M\mu}$, and $P_i = \dfrac{r\left(B + \mu N_i^2\right)(K - N_i)}{K\alpha}$. The equilibrium point E_i is locally asymptotically stable if $2\mu K N_i - B - 3\mu N_i^2 < 0$ and $B - \mu N_i^2 - 2\mu qmAN_i > 0$. Conversely, E_i is a saddle point if $B - \mu N_i^2 - 2\mu qmAN_i < 0$.*

Proof. At the interior equilibrium point $E_i = (N_i, P_i)$, the Jacobian matrix is given by

$$J(E_i) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix}, \tag{6}$$

where

$$a_{11} = \frac{rN_i\left(2\mu K N_i - B - 3\mu N_i^2\right)}{K\left(B + \mu N_i^2\right)},$$
$$a_{12} = -\frac{\alpha N_i}{B + \mu N_i^2},$$
$$a_{21} = \frac{\zeta P_i\left(B - \mu N_i^2 - 2\mu qmAN_i\right)}{\left(B + \mu N_i^2\right)^2}.$$

Since

$$N_i > 0, \qquad \alpha > 0, \qquad B + \mu N_i^2 > 0,$$

we have $a_{12} < 0$. From the Jacobian matrix (6), the trace and determinant are respectively given by

$$\operatorname{tr}\left(J(E_i)\right) = a_{11},$$

and

$$\det\left(J(E_i)\right) = -a_{12}a_{21}.$$

By the Routh–Hurwitz criterion, the interior equilibrium point E_i is locally asymptotically stable if

$$\operatorname{tr}\left(J(E_i)\right) < 0$$

and

$$\det\left(J(E_i)\right) > 0.$$

Since $a_{12} < 0$, we have

$$\det\left(J(E_i)\right) > 0$$

if and only if $a_{21} > 0$. From the expression for a_{21}, since

$$\zeta > 0, \qquad P_i > 0, \qquad \left(B + \mu N_i^2\right)^2 > 0,$$

it follows that $a_{21} > 0$ if and only if

$$B - \mu N_i^2 - 2\mu qmAN_i > 0.$$

Meanwhile,

$$\operatorname{tr}\left(J(E_i)\right) < 0$$

if and only if

$$2\mu KN_i - B - 3\mu N_i^2 < 0.$$

Therefore, E_i is locally asymptotically stable if

$$2\mu KN_i - B - 3\mu N_i^2 < 0$$

and

$$B - \mu N_i^2 - 2\mu qmAN_i > 0.$$

Conversely, if

$$B - \mu N_i^2 - 2\mu qmAN_i < 0,$$

then $a_{21} < 0$. Since $a_{12} < 0$, we obtain

$$\det\left(J(E_i)\right) < 0.$$

Consequently, the eigenvalues have opposite signs, and E_i is a saddle point. □

Biologically, Theorem 2 shows that the state in which both prey and predator populations are absent is unstable because the prey population still has the ability to grow when its density is low. Theorem 3 indicates that the stability of the predator-free equilibrium is determined by the predator's ability to invade the system when the prey population is at its carrying capacity. In this case, the alternative food quality parameter q directly affects the second eigenvalue at E_1. A larger value of q increases the contribution of alternative food to predator growth, thereby increasing the possibility of predator invasion into the ecosystem. Theorem 4 shows that the stability of the interior equilibrium is influenced by two main mechanisms. The first condition is related to the prey growth response and the effect of group defense, while the second condition is associated with the sensitivity of predator growth to changes in prey density. The presence of the parameter q in the second condition confirms that alternative food quality not only affects the existence of the interior equilibrium but also determines the stability of coexistence between prey and predator populations.

3.4 Forward Bifurcation

This section discusses the change in stability of the predator-free equilibrium point $E_1 = (K, 0)$ when the alternative food quality parameter q is varied. Based on Theorem 3, the stability of E_1 is determined by the eigenvalue

$$\lambda_2 = \frac{\zeta m A}{B + \mu K^2}(q - q_c),$$

where

$$q_c = \frac{M\left(B + \mu K^2\right) - \zeta K}{\zeta m A}.$$

Since

$$\frac{\zeta m A}{B + \mu K^2} > 0,$$

the sign of $\lambda_2(q)$ is determined by the sign of $q - q_c$. Hence, the equilibrium point E_1 is locally asymptotically stable if $q < q_c$, becomes a saddle point if $q > q_c$, and is nonhyperbolic if $q = q_c$.

The critical bifurcation value is therefore given by $q = q_c$. This value serves as a threshold for the predator's ability to invade the ecosystem. When $q < q_c$, the quality of alternative food is not sufficient to support predator growth, so the predator-free equilibrium remains stable. Conversely, when $q > q_c$, the quality of alternative food is high enough to support predator growth, causing the predator-free equilibrium to lose stability and allowing a positive interior equilibrium branch to emerge. Mathematically, this transition corresponds to a forward bifurcation, as stated in Theorem 5.

Theorem 5. *Let*

$$q_c = \frac{M\left(B + \mu K^2\right) - \zeta K}{\zeta m A}. \tag{7}$$

If $\zeta > 2M\mu K$, then model (3) undergoes a forward bifurcation at the predator-free equilibrium point $E_1 = (K, 0)$ when $q = q_c$. More specifically, if $q < q_c$, then the predator-free equilibrium point $E_1 = (K, 0)$ is locally asymptotically stable. If $q > q_c$, then $E_1 = (K, 0)$ loses its stability and becomes a saddle point. In a neighborhood of $q = q_c$, a positive interior equilibrium branch emerges, representing the coexistence of prey and predator populations.

Proof. Based on Theorem 3, the eigenvalue that determines the stability of the predator-free equilibrium point $E_1 = (K, 0)$ is

$$\lambda_2(q) = \frac{\zeta(K + qmA)}{B + \mu K^2} - M.$$

This eigenvalue can be written as

$$\lambda_2(q) = \frac{\zeta m A}{B + \mu K^2}(q - q_c),$$

where

$$q_c = \frac{M\left(B + \mu K^2\right) - \zeta K}{\zeta m A}.$$

Thus, at $q = q_c$, we have $\lambda_2(q_c) = 0$. This means that one eigenvalue of the Jacobian matrix at E_1 is zero. Hence, E_1 is nonhyperbolic.

Next, to verify the transversality condition, we compute

$$\frac{d\lambda_2}{dq} = \frac{\zeta m A}{B + \mu K^2}.$$

Since all parameters are positive and $B + \mu K^2 > 0$, it follows that

$$\frac{d\lambda_2}{dq} > 0.$$

Therefore, as q passes through q_c, the eigenvalue $\lambda_2(q)$ crosses zero from negative to positive. Consequently, for $q < q_c$, we have $\lambda_2(q) < 0$, so E_1 is locally asymptotically stable. Conversely, for $q > q_c$, we have $\lambda_2(q) > 0$, so E_1 becomes a saddle point.

It remains to show that the interior equilibrium branch emerges from E_1 when $q = q_c$. At the interior equilibrium, the prey component N satisfies

$$M\mu N^2 - \zeta N + MB - \zeta qmA = 0.$$

Substituting $q = q_c$ into this equation gives

$$M\mu N^2 - \zeta N + MB - \zeta mA\left(\frac{M\left(B + \mu K^2\right) - \zeta K}{\zeta mA}\right) = 0.$$

After simplification, we obtain

$$M\mu N^2 - \zeta N - M\mu K^2 + \zeta K = 0.$$

This equation can be factored as

$$(N - K)\left[M\mu(N + K) - \zeta\right] = 0.$$

Hence, one interior-branch root at $q = q_c$ is $N = K$. For $N = K$, the predator component is

$$P = \frac{r\left(B + \mu K^2\right)(K - K)}{K\alpha} = 0.$$

Thus, at $q = q_c$, the interior equilibrium branch intersects the predator-free equilibrium point $E_1 = (K, 0)$.

To determine the direction in which the interior branch emerges, define

$$H(N, q) = M\mu N^2 - \zeta N + MB - \zeta qmA.$$

At $(N, q) = (K, q_c)$, we have

$$H(K, q_c) = 0.$$

Moreover,

$$\frac{\partial H}{\partial N}(K, q_c) = 2M\mu K - \zeta.$$

If $\zeta \neq 2M\mu K$, then, by the implicit function theorem, there exists a branch $N = N(q)$ near $q = q_c$. The derivative of this branch is

$$\frac{dN}{dq} = -\frac{H_q}{H_N} = \frac{\zeta mA}{2M\mu K - \zeta}.$$

For a forward bifurcation, the positive interior branch must emerge for $q > q_c$. Since

$$P = \frac{r\left(B + \mu N^2\right)(K - N)}{K\alpha},$$

the condition $P > 0$ is equivalent to $N < K$. If $\zeta > 2M\mu K$, then

$$2M\mu K - \zeta < 0,$$

so

$$\frac{dN}{dq} < 0.$$

This means that when q increases beyond q_c, the branch moves toward $N(q) < K$. Consequently, $P(q) > 0$, and a positive interior equilibrium branch emerges for $q > q_c$. Therefore, as q passes through q_c, the predator-free equilibrium point loses stability, and a positive interior equilibrium branch emerges from E_1. Hence, model (3) undergoes a forward bifurcation at $q = q_c$. □

3.5 Numerical Simulation

Numerical simulations are carried out to confirm the analytical results obtained in the previous sections, particularly regarding the role of the alternative food quality parameter q in changing the stability of equilibrium points. The main focus of the simulations is to show that when $q < q_c$, the predator-free equilibrium point $E_1 = (K, 0)$ is stable, whereas when $q > q_c$, this point loses its stability and the solution moves toward an interior equilibrium representing the coexistence of prey and predator populations.

The parameter values used in the simulations are

$$r = 1, \quad K = 10, \quad \alpha = 1, \quad \beta = 1, \quad \kappa = 0.5, \quad m = 1, \quad A = 2, \quad \mu = 0.01, \quad \zeta = 1, \quad \delta = 3.5, \quad \theta = 0.1.$$

Based on these parameter values, we obtain $B = \beta + \kappa mA = 2$ and $M = \delta + \theta mA = 3.7$. Substituting these parameter values into the bifurcation threshold in equation (7) gives

$$q_c = 0.55,$$

which serves as the threshold value of alternative food quality determining the change in stability of the predator-free equilibrium point.

The numerical simulations are divided into three cases: the stable E_1 case, which represents predator extinction when the quality of alternative food is low; the stable interior equilibrium case, which represents coexistence of the two populations when the quality of alternative food is high; and the bifurcation diagram with respect to the parameter q, which represents the transition from predator extinction to coexistence of the two populations.

3.5.1 Low Alternative Food Quality: Stability of the Predator-Free Equilibrium

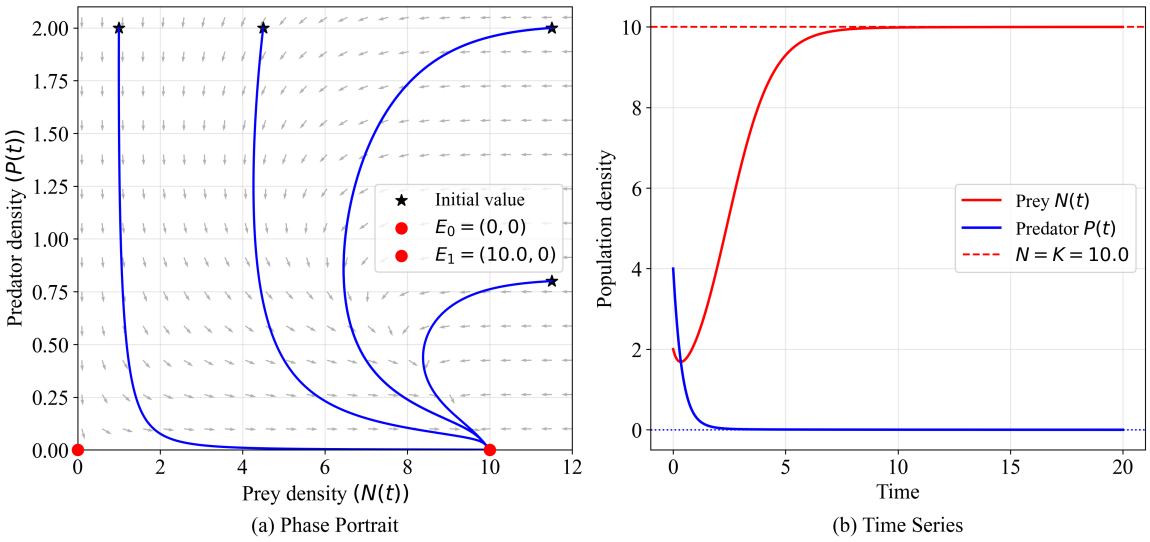
The first case is used to illustrate the condition in which the quality of alternative food is still low. In this simulation, we choose

$$q = 0.30 < q_c = 0.55.$$

For this value, the second eigenvalue at the predator-free equilibrium point $E_1 = (K, 0)$ is

$$\lambda_2(E_1) = \frac{\zeta(K + qmA)}{B + \mu K^2} - M = \frac{10 + 0.30(2)}{3} - 3.7 \approx -0.1667.$$

Since $\lambda_1 = -r = -1 < 0$ and $\lambda_2(E_1) < 0$, the predator-free equilibrium point $E_1 = (10, 0)$ is locally asymptotically stable. The numerical simulation for this case is shown in Figure 1.



(a) Phase Portrait (b) Time Series

Figure 1: Numerical simulation for $q = 0.30 < q_c$: (a) phase portrait; (b) time series of prey and predator populations.

Figure 1(a) shows that two equilibrium points exist, namely the trivial equilibrium $E_0 = (0, 0)$ and the predator-free equilibrium $E_1 = (10, 0)$. Several solution trajectories with different initial conditions move toward $E_1 = (10, 0)$, confirming that the predator-free equilibrium is locally asymptotically stable. Figure 1(b)

illustrates the population dynamics over time, where the prey population $N(t)$ increases and stabilizes when it reaches the carrying capacity $K = 10$, while the predator population $P(t)$ decreases toward zero. This indicates that the prey can persist up to the environmental carrying capacity, whereas the predator becomes extinct in the long term.

Biologically, the condition $q = 0.30$ represents a situation in which alternative food is available but has low quality. Although the predator still has access to alternative food, its effective contribution to predator growth is not sufficient to compensate for the effective mortality $\delta + \theta mA$. As a result, the predator is unable to maintain its population and eventually becomes extinct. When the predator becomes extinct, predation pressure on the prey disappears, allowing the prey population to grow toward the environmental carrying capacity. Thus, low alternative food quality causes the system to end in a predator-free state.

3.5.2 High Alternative Food Quality: Stable Coexistence of Prey and Predator

The second case is used to illustrate the condition in which the quality of alternative food is sufficiently high to support predator persistence. In this simulation, we choose

$$q = 0.80 > q_c = 0.55.$$

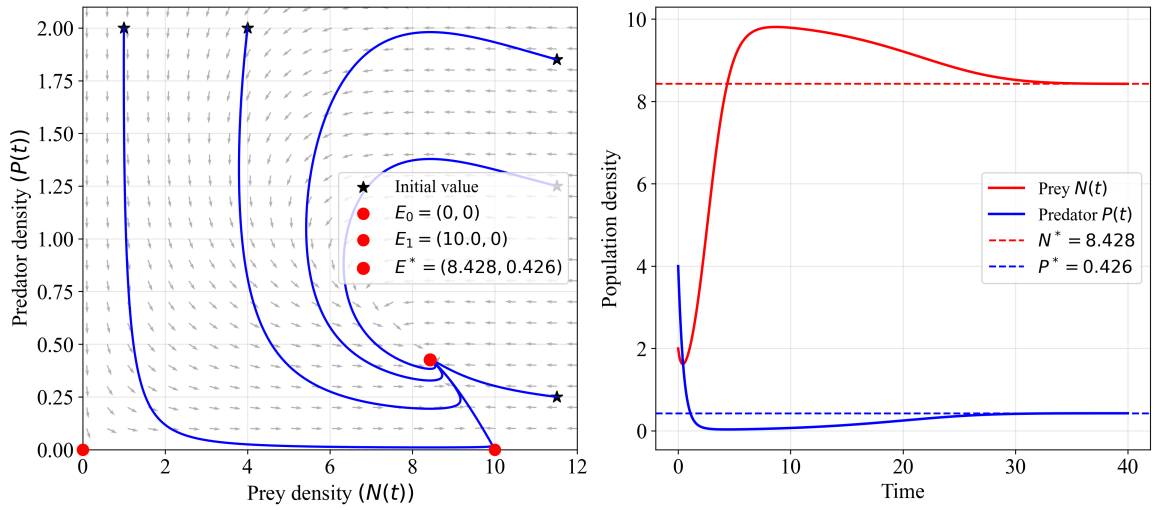
For this value, the second eigenvalue at the predator-free equilibrium point is

$$\lambda_2(E_1) = \frac{\zeta(K + qmA)}{B + \mu K^2} - M = \frac{10 + 0.80(2)}{3} - 3.7 \approx 0.1667.$$

Since $\lambda_2(E_1) > 0$, the equilibrium point $E_1 = (10, 0)$ loses its stability and becomes a saddle point. At the same time, there exists a biologically feasible positive interior equilibrium. Based on the numerical calculation, we obtain

$$E^* = (N^*, P^*) \approx (8.428, 0.426).$$

The numerical simulation for this case is shown in Figure 2.



(a) Phase Portrait (b) Time Series

Figure 2: Numerical simulation for $q = 0.80 > q_c$: (a) phase portrait; (b) time series of prey and predator populations.

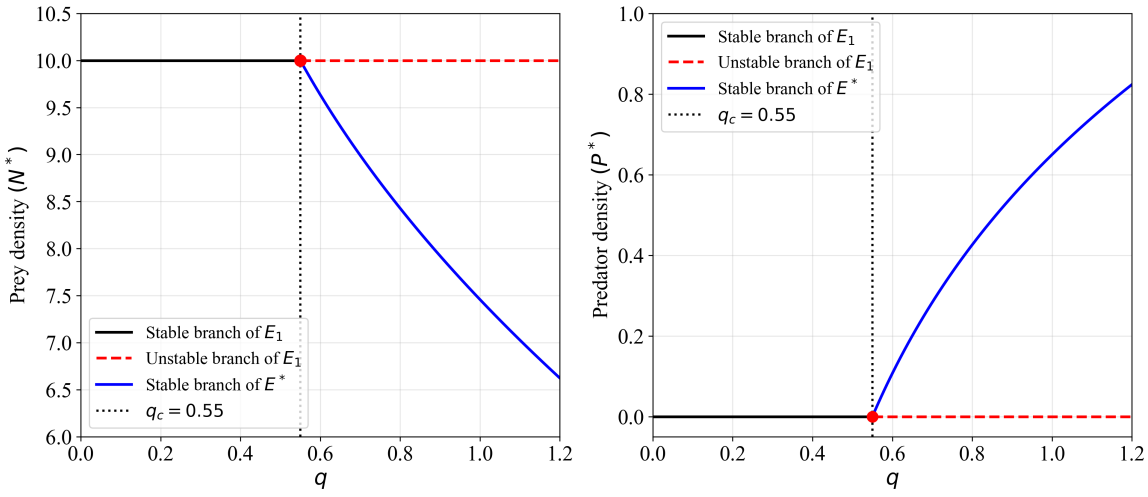
Figure 2(a) shows that three equilibrium points exist, namely the trivial equilibrium $E_0 = (0, 0)$, the predator-free equilibrium $E_1 = (10, 0)$, and the interior equilibrium $E^* \approx (8.428, 0.426)$. Several solution trajectories with different initial conditions move toward the interior equilibrium E^*. This confirms that when $q = 0.80 > q_c$, the predator-free equilibrium loses its stability, along with the emergence of a locally asymptotically stable interior equilibrium. In other words, the system does not end in predator extinction, but instead approaches a coexistence state between prey and predator populations. Figure 2(b) illustrates the population dynamics over time. The prey population $N(t)$ initially increases sharply, then decreases gradually and eventually stabilizes around $N^* \approx 8.428$. Meanwhile, the predator population $P(t)$ initially decreases sharply toward zero, then

gradually increases and stabilizes around $P^* \approx 0.426$. This indicates that both prey and predator populations are able to persist in the long term.

Biologically, the condition $q = 0.80$ represents a situation in which alternative food is available with sufficiently high quality. The effective contribution of alternative food to predator growth is sufficient to compensate for the effective mortality $\delta + \theta m A$. As a result, the predator does not go extinct, but is able to maintain its population in the system. At the same time, the presence of the predator continues to exert predation pressure on the prey, so the prey population cannot reach the maximum carrying capacity $K = 10$, but instead stabilizes at a lower value. Thus, high alternative food quality allows the system to end in stable coexistence between prey and predator populations.

3.5.3 Alternative Food Quality Threshold: Transition from Predator Extinction to Coexistence

The third simulation is conducted to illustrate the change in equilibrium stability when the alternative food quality parameter q is varied. The forward bifurcation diagram in Figure 3 shows the equilibrium branches of prey and predator populations with respect to changes in q. The main purpose of this simulation is to emphasize that q acts as a threshold parameter governing the transition from a predator-free state to coexistence.



(a) Prey population (b) Predator population

Figure 3: Forward bifurcation diagram with respect to the parameter q: (a) changes in prey density N^*; (b) changes in predator density P^*.

Figure 3 shows the forward bifurcation diagram with respect to the alternative food quality parameter q, with the critical point $q_c = 0.55$. Figure 3(a) illustrates the change in prey density N^*, while Figure 3(b) illustrates the change in predator density P^*. In Figure 3, the black line represents the stable branch of the predator-free equilibrium E_1, the red dashed line represents the unstable branch of E_1, the blue line represents the stable branch of the interior equilibrium E^*, and the vertical dotted line represents the critical bifurcation value q_c.

In Figure 3(a), when $q < q_c$, the stable branch lies at $N^* = 10$, indicating that the prey population remains at the environmental carrying capacity when the predator becomes extinct. When q passes through q_c, the predator-free equilibrium branch loses its stability and the stable interior equilibrium branch emerges. For $q > q_c$, the value of N^* on the interior branch decreases as q increases. This indicates that the higher the quality of alternative food, the more capable the predator is of persisting and exerting stronger predation pressure on the prey, causing the prey density at equilibrium to become lower than its carrying capacity. In Figure 3(b), when $q < q_c$, the stable branch lies at $P^* = 0$, indicating predator extinction. At $q = q_c$, a stability change occurs. For $q > q_c$, a positive predator branch emerges and increases as q increases. This shows that once the quality of alternative food exceeds the threshold value, the predator is able to maintain its population and coexist with the prey.

Biologically, Figure 3 confirms that $q_c = 0.55$ is the alternative food quality threshold for predator persistence. If $q < q_c$, alternative food is available but its quality is not sufficient to support predator growth, so the system ends in predator extinction. Conversely, if $q > q_c$, the alternative food has sufficiently high quality to compensate for the predator's effective mortality. As a result, the predator can invade the system, and the

system moves toward stable coexistence. Thus, increasing the quality of alternative food triggers a transition from predator extinction to coexistence through a forward bifurcation.

4. Conclusion

This study develops a predator–prey model by incorporating logistic prey growth, prey group defense, alternative food, and predator ecological costs. The novelty of the model lies in the introduction of an alternative food quality parameter, which distinguishes the availability of alternative food from its effectiveness in supporting predator growth.

The analytical results show that the model has three types of equilibrium points, namely the trivial equilibrium, the predator-free equilibrium, and the interior equilibrium. The trivial equilibrium is always unstable. The predator-free equilibrium is stable when the quality of alternative food is below its critical value, but loses stability when the alternative food quality exceeds this threshold. Under this condition, a positive interior equilibrium emerges and may become stable, causing the system to move toward coexistence between prey and predator populations. The bifurcation analysis shows that alternative food quality acts as a critical threshold for predator persistence. When the quality of alternative food is low, the predator cannot persist and the system approaches a predator-free state. Conversely, when the alternative food quality is sufficiently high, the predator can invade the system and form stable coexistence. Thus, increasing the quality of alternative food can trigger a transition from predator extinction to coexistence through a forward bifurcation.

Numerical simulations confirm these analytical results. Under low alternative food quality, the solution approaches the predator-free state. Under high alternative food quality, the solution approaches a stable interior equilibrium. The bifurcation diagram also shows a stability transition from the predator-free branch to the interior branch when the alternative food quality exceeds its critical value.

Biologically, these results emphasize that the availability of alternative food alone is not sufficient to guarantee predator persistence. The quality of alternative food must exceed a certain threshold to support predator growth and allow coexistence. Future studies may extend this model by considering dynamic alternative food, seasonal variation, or the possibility of other bifurcations under different parameter regimes.

Author Contributions

Resmawan: Conceptualization, methodology, validation, formal analysis, supervision, writing—review and editing. **Binti Mualifatul Rosydah**: Methodology, software, formal analysis, investigation, writing—original draft preparation, visualization. **Rizka Putri Handayani**: Validation, investigation, resources, writing—review and editing, visualization.

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Conflicts of Interest

The authors declare no conflict of interest.

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