An LMI Approach to H_∞ Performance Analysis of Continuous-Time Systems with Two Additive Time-Varying Delays

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Abstract. This paper investigates the problem of H_∞ performance analysis for continous–time systems with two additive time-varying delays in the state. Our objective is focused on stability analysis of a continuous system with two time-varying delays with an H_∞ disturbance attenuation level γ. By exploiting Lyapunov-Krasovski functional and introducing free weighting matrix variables, LMI stability condition have been derived.

Keywords: *H_∞ performance analysis; Linear Matrix Inequality (LMI); time delay systems.*

1 Introduction

Time delay is the property of a physical system by which the response to an applied force (action) is delayed in its effect. When information or energy is physically transmitted from one place to another, there is a delay associated with the transmission [1]. It is well known that the presence of time-delay is a source of instability [2]. Xia, *et al.* [3] presents some basic theories of stability and synthesis of systems with time-delay, in the form of $\dot{x}(t) = Ax(t) + A_d x(t-\tau(t))$, where $\tau(t)$ represents time-varying delay. Wu, *et al.* [4] presents a method referred to as the free-weighting-matrix (FWM) approach for the stability analysis and control synthesis of various classes of time-delay systems. In [5], a new model for time delay systems is proposed, that is $\dot{x}(t) = Ax(t) + A_d x(t-\tau_1(t)-\tau_2(t))$. The new model is motivated by practical situation in Networked Control Systems (NCSs), where $\tau_1(t)$ is the time-delay from sensor to the controller and $\tau_2(t)$ is the time-delay from controller to the actuator.

Motivated by stability condition for system with two delays in the state, derived in [6], in this paper we investigate conditions under which the continuous system with two time-varying delays in the state is asymptotically stable with an H_∞ disturbance attenuation level γ. It is well known in systems and control

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community that H_∞-norm constraint can be used to provide a prespecified disturbance attenuation level, and alternatively to analyze robust stability of dynamical system under unstructured uncertainty. By exploiting Lyapunov-Krasovski functional from [6] and introducing free weighting matrix variables, the stability condition for the system is derived by using linear matrix inequality (LMI) techniques.

Notation. The notation $X > 0$ denotes a symmetric positive definite, asterisk (*) represents the elements of symmetric term in the symmetric block matrix. The superscripts "T" and "-1" represent the transpose and inverse matrix, respectively. $L_2[0,\infty)$ is the space of square integrable functions on $[0,\infty)$.

2 Problem definition

As stated previously, systems with two delays in the state can be found in the Networked Control System (NCS), shown in Figure 1 [5]. In NCS, the physical plant, sensor, controller and actuator are located at different locations and hence the signals among those components are transmitted over network media.

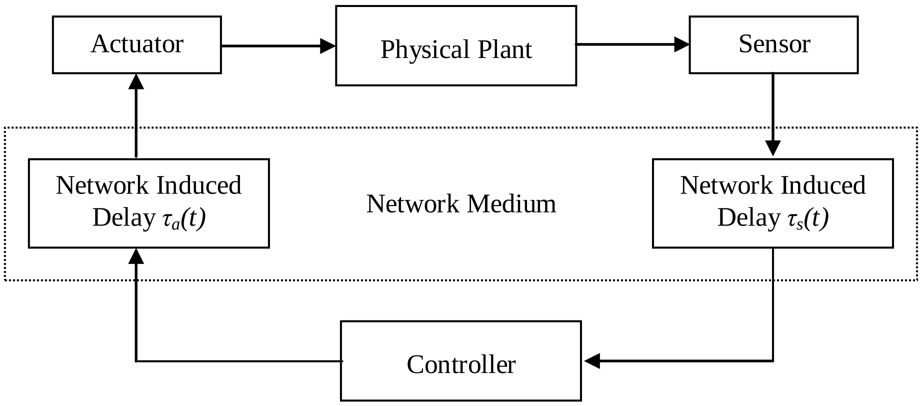

Figure 1 Networked control system [5].

We can see in Figure 1 that there are two delays, $\tau_s(t)$ represents the delay of data transmission from sensor to controller while $\tau_a(t)$ represents the delay from controller to actuator. The properties of these two delays may not be identical due to the network transmission condition, and hence it is not reasonable to combine them together [5]. Based on this observation, Lam, *et al.* [5] proposed new model for time delay systems, described by $\dot{x}(t) = Ax(t) + A_d x(t - \tau_1(t) - \tau_2(t))$. Based on such a system representation, Lam, *et al.* [5] derived the stability condition. Gao, *et al.* [7] presented a new stability condition and investigated the problem of H_∞ performance analysis. Dey, *et al.* [6] constructed a new Lyapunov-Krasovskii functional in obtaining the stability condition for such system, and provided less conservative delay upper bound, as compared to the conditions in [5,7,8]. By using Lyapunov-Krasovskii functional in [6], in the present paper we investigate the problem of H_∞ performance analysis for continuous–time systems with two additive time-varying delays in the state.

Consider the following system with two additive time varying delays in the state [7],

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x\big(t-\tau_1(t)-\tau_2(t)\big) + Ew(t), \\ y(t) &= Cx(t) + C_d x\big(t-\tau_1(t)-\tau_2(t)\big) + Fw(t), \\ x(t) &= \phi(t), \qquad t \in [-\bar{\tau}, 0] \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$ is the state vector; $y(t) \in \Re^p$ is the output vector; $\tau_1(t)$ and $\tau_2(t)$ represent two delays in the state; $\phi(t)$ is the initial condition on the segment $[-\bar{\tau}, 0]$; $w(t) \in \Re^l$ is the disturbance input which belongs to $L_2[0,\infty)$; A, A_d, E, C, C_d, and F are known system matrices with appropriate dimension. For system in Eq. (1), it is assumed that [6,7],

$$0 \le \tau_1(t) \le \bar{\tau}_1 < \infty,\ \dot{\tau}_1 \le d_1 < \infty,\ 0 \le \tau_2(t) \le \bar{\tau}_2 < \infty,\ \ \dot{\tau}_2 \le d_2 < \infty, \tag{2}$$

and $\bar{\tau} = \bar{\tau}_1 + \bar{\tau}_2, \quad d = d_1 + d_2$

Our objective is to investigate whether the continuous-time system with two time-varying delays in the state is asymptotically stable with an H_∞ disturbance attenuation level γ.

Definition 1. If there exist positive definite Lyapunov function $V(x,t)$ such that the derivatives with respect to time t ($w = 0$) satisfies $\dot{V}(x,t) < 0$, then system (1) is said to be asymptotically stable.

Lemma 1 [9]. For any $z, y \in \Re^n$ and for any symmetric positive definite matrix $X \in \Re^{n \times n}$,

$$-2z^T y \le z^T X^{-1} z + y^T X y$$

Lemma 2 [10]**.** Schur Complement. Schur's formula says that the following statements are equivalent:

i. $\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} < 0$

ii. $\begin{array}{l} \Phi_{22} < 0 \\ \Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{12}^T < 0 \end{array}$

3 Main Result

The Main result of the present paper is stated in the following theorem.

Theorem 1. The continuous-time system with two additive time-varying delays in the state (1) satisfying (2) either

(i) asymptotically stable with $w = 0$, or

(ii) stable with H_∞ disturbance attenuation level γ ($w \neq 0$)

if there exist matrices $P = P^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, $Q_3 = Q_3^T > 0$, $R_1 = R_1^T > 0$, $R_2 = R_2^T > 0$, $R_3 = R_3^T > 0$ and G_i, L_i, M_i, N_i, $i = 1,..., 4$ are free matrices with $Q_2 \geq Q_3$ satisfying,

$$\begin{bmatrix} \Psi_1 + \Psi_2 + \Psi_2^T + \Psi_3 + \Psi_6^T\Psi_6 & \Psi_5 \\ * & \Xi_5 \end{bmatrix} < 0 \tag{3}$$

where,

$$\Psi_1 = \begin{bmatrix} Q_1 + Q_2 & 0 & 0 & P & 0 \\ * & -(1-d_1)(Q_2 - Q_3) & 0 & 0 & 0 \\ * & * & -(1-d_1-d_2)(Q_1+Q_3) & 0 & 0 \\ * & * & * & \bar{\tau}R_1 + \bar{\tau}_1R_2 + \bar{\tau}_2R_3 & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix}$$

$\Psi_2 = \Psi_{21}\Psi_{22}$, $\Psi_{21} = \begin{bmatrix} A & 0 & A_d & -I & E \end{bmatrix}^T$,

$\Psi_{22} = \begin{bmatrix} G_1^T & G_2^T & G_3^T & G_4^T & 0 \end{bmatrix}$,

$$\Psi_3 = \begin{bmatrix} L_1 + L_1^T + M_1 + M_1^T & L_2^T - M_1 + M_2^T + N_1 & -L_1 + L_3^T + M_3^T - N_1 & L_4^T + M_4^T & 0 \\ * & -M_2 - M_2^T + N_2 + N_2^T & -L_2 - M_3^T - N_2 + N_3^T & -M_4^T + N_4^T & 0 \\ * & * & -L_3 - L_3^T - N_3 - N_3^T & -L_4^T - N_4^T & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\Psi_5 = \begin{bmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \\ L_4 & M_4 & N_4 \\ 0 & 0 & 0 \end{bmatrix},\ \Psi_6 = \begin{bmatrix} C & 0 & C_d & 0 & F \end{bmatrix},\ \Xi_5 = \mathrm{diag}\left\{-\bar{\tau}^{-1}R_1, -\bar{\tau}_1^{-1}R_2, -\bar{\tau}_2^{-1}R_3\right\}$$

Proof.

Define a Lyapunov-Krasovski functional as in [6],

$$V(t) = V_1(t) + V_2(t) + V_3(t)$$

$$V_1(t) = x^T(t)Px(t), \tag{4}$$

$$V_2(t) = \int_{t-\tau_1(t)-\tau_2(t)}^{t} x^T(s)Q_1x(s)ds + \int_{t-\tau_1(t)}^{t} x^T(s)Q_2x(s)ds + \int_{t-\tau_1(t)-\tau_2(t)}^{t-\tau_1(t)} x^T(s)Q_3x(s)ds \tag{5}$$

$$V_3(t) = \int_{t-\bar{\tau}_1-\bar{\tau}_2}^{t}\int_{\theta}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta + \int_{t-\bar{\tau}_1}^{t}\int_{\theta}^{t} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta + \int_{t-\bar{\tau}_1-\bar{\tau}_2}^{t-\bar{\tau}_1}\int_{\theta}^{t} \dot{x}^T(s)R_3\dot{x}(s)dsd\theta \tag{6}$$

The time derivative of *V(t)* satisfying condition (2) is given by (as done in [5])

$$\dot{V}_1(t) = 2x^T(t)P\dot{x}(t), \tag{7}$$

$$\begin{aligned}\dot{V}_2(t) \le{}& x^T(t)(Q_1 + Q_2)x(t) - (1-d_1)x^T(t-\tau_1(t))(Q_2 - Q_3)x(t-\tau_1(t)) \\ & -(1-d_1-d_2)x^T(t-\tau(t))(Q_1 + Q_3)x(t-\tau(t))\end{aligned} \tag{8}$$

$$\begin{aligned}\dot{V}_3(t) \le{}& \dot{x}^T(t)(\bar{\tau}R_1 + \bar{\tau}_1R_2 + \bar{\tau}_2R_3)\dot{x}(t) - \int_{t-\tau(t)}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds \\ & - \int_{t-\tau_1(t)}^{t} \dot{x}^T(s)R_2\dot{x}(s)ds - \int_{t-\tau(t)}^{t-\tau_1(t)} \dot{x}^T(s)R_3\dot{x}(s)ds\end{aligned} \tag{9}$$

Now, introducing any free matrices G_i, $i=1,2,3,4$, one may write

$$\begin{aligned}\Sigma = 2\left[x^T(t)G_1 + x^T(t-\tau_1(t))G_2 + x^T(t-\tau(t))G_3 + \dot{x}^T(t)G_4\right] \\ \times\left[-\dot{x}(t) + Ax(t) + A_dx(t-\tau(t)) + Ew(t)\right] = 0\end{aligned} \tag{10}$$

Simplifying Eq. (10), we get

$$\bar{\xi}^T(t)\begin{bmatrix} G_1A+A^TG_1^T & A^TG_2^T & G_1A_d+A^TG_3^T & -G_1+A^TG_4^T & G_1E \\ * & 0 & G_2A_d & -G_2 & G_2E \\ * & * & G_3A_d+A_d^TG_3^T & -G_3+A_d^TG_4^T & G_3E \\ * & * & * & -G_4-G_4^T & G_4E \\ * & * & * & * & 0 \end{bmatrix}\bar{\xi}(t)=0 \tag{11}$$

where $\bar{\xi}(t)=\left[x^T(t) \quad x^T(t-\tau_1(t)) \quad x^T(t-\tau(t)) \quad \dot{x}^T(t) \quad w(t)\right]^T$.

Now, to eliminate integral terms in Eq. (9), we can use the Newton-Leibniz formula, and introducing free matrices L_i, $i = 1, 2, 3, 4$, we get

$$\begin{aligned} &2\left[x^T(t)L_1+x^T(t-\tau_1(t))L_2+x^T(t-\tau(t))L_3+\dot{x}^T(t)L_4\right] \\ &\times\left[x(t)-x(t-\tau(t))-\int_{t-\tau(t)}^{t}\dot{x}(s)ds\right]=0 \end{aligned} \tag{12}$$

Simplifying Eq. (12), we obtain

$$\bar{\xi}^T(t)\begin{bmatrix} L_1+L_1^T & L_2^T & -L_1+L_3^T & L_4^T & 0 \\ * & 0 & -L_2 & 0 & 0 \\ * & * & -L_3-L_3^T & -L_4^T & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}\bar{\xi}(t)+\int_{t-\tau(t)}^{t}-2\bar{\xi}^T(t)\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ 0 \end{bmatrix}\dot{x}(s)ds=0 \tag{13}$$

Applying Lemma 1 on the last term of Eq. (13), we get

$$\int_{t-\tau(t)}^{t}-2\bar{\xi}^T(t)\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ 0 \end{bmatrix}\dot{x}(s)ds\le\tau(t)\bar{\xi}^T(t)\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ 0 \end{bmatrix}R_1^{-1}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ 0 \end{bmatrix}^T\bar{\xi}(t)+\int_{t-\tau(t)}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds \tag{14}$$

Substituting Eq. (14) in the last term of Eq. (13) and with little manipulation, we get

$$-\int_{t-\tau(t)}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds\le\bar{\xi}^T(t)\left\{\begin{bmatrix} L_1+L_1^T & L_2^T & -L_1+L_3^T & L_4^T & 0 \\ * & 0 & -L_2 & 0 & 0 \\ * & * & -L_3-L_3^T & -L_4^T & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}+\bar{\tau}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ 0 \end{bmatrix}R_1^{-1}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ 0 \end{bmatrix}^T\right\}\bar{\xi}(t) \tag{15}$$

We can remove the last two terms of Eq. (9) (integral terms) using similar way as done for Eqs. (12) – (15), to obtain

$$-\int_{t-\tau_1(t)}^{t}\dot{x}^T(s)R_2\dot{x}(s)ds \le \bar{\xi}^T(t)\left\{\begin{bmatrix} M_1+M_1^T & -M_1+M_2^T & M_3^T & M_4^T & 0 \\ * & -M_2-M_2^T & -M_3^T & -M_4^T & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix} + \bar{\tau}_1\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ 0 \end{bmatrix} R_2^{-1}\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ 0 \end{bmatrix}^T\right\}\bar{\xi}(t) \tag{16}$$

$$-\int_{t-\tau(t)}^{t-\tau_1(t)}\dot{x}^T(s)R_3\dot{x}(s)ds \le \bar{\xi}^T(t)\left\{\begin{bmatrix} 0 & N_1 & -N_1 & 0 & 0 \\ * & N_2+N_2^T & -N_2+N_3^T & N_4^T & 0 \\ * & * & -N_3-N_3^T & -N_4^T & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix} + \bar{\tau}_2\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ 0 \end{bmatrix} R_3^{-1}\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ 0 \end{bmatrix}^T\right\}\bar{\xi}(t) \tag{17}$$

where M_i, $i = 1, 2, 3, 4$ and N_i, $i = 1, 2, 3, 4$ are free matrices.

Substituting Eqs. (15), (16) and (17) into Eq. (9), we get

$$\dot{V}_3(t) \le \dot{x}^T(t)\left(\bar{\tau}R_1+\bar{\tau}_1R_2+\bar{\tau}_2R_3\right)\dot{x}(t)+\bar{\xi}^T(t)\left\{\dot{V}_{31}+\dot{V}_{32}+\dot{V}_{33}+\dot{V}_{34}\right\}\bar{\xi}(t) \tag{18}$$

where

$$\dot{V}_{31} = \begin{bmatrix} L_1+L_1^T+M_1+M_1^T & L_2^T-M_1+M_2^T+N_1 & -L_1+L_3^T+M_3^T-N & L_4^T+M_4^T & 0 \\ * & -M_2-M_2^T+N_2+N_2^T & -L_2-M_3^T-N_2+N_3^T & -M_4^T+N_4^T & 0 \\ * & * & -L_3-L_3^T-N_3-N_3^T & -L_4^T-N_4^T & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}$$

$$\dot{V}_{32} = \bar{\tau}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ 0 \end{bmatrix} R_1^{-1}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ 0 \end{bmatrix}^T, \quad \dot{V}_{33} = \bar{\tau}_1\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ 0 \end{bmatrix} R_2^{-1}\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ 0 \end{bmatrix}^T, \quad \dot{V}_{34} = \bar{\tau}_2\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ 0 \end{bmatrix} R_3^{-1}\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ 0 \end{bmatrix}^T$$

Then, from Eqs. (7), (8), (11) and (18) we have

$$\dot{V}(t) = \dot{V}_1(t)+\dot{V}_2(t)+\dot{V}_3(t)+\Sigma$$

$$\dot{V}(t) \le \bar{\xi}^T(t)\left\{\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} \\ * & * & \Omega_{33} & \Omega_{34} & \Omega_{35} \\ * & * & * & \Omega_{44} & \Omega_{45} \\ * & * & * & * & \Omega_{55} \end{bmatrix} + \bar{\tau}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ 0 \end{bmatrix} R_1^{-1}\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ 0 \end{bmatrix}^T + \bar{\tau}_1\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ 0 \end{bmatrix} R_2^{-1}\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ 0 \end{bmatrix}^T + \bar{\tau}_2\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ 0 \end{bmatrix} R_3^{-1}\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ 0 \end{bmatrix}^T\right\}\bar{\xi}(t) \tag{19}$$

where,

$\Omega_{11} = Q_1 + Q_2 + G_1 A + A^T G_1^T + L_1 + L_1^T + M_1 + M_1^T$,
$\Omega_{12} = A^T G_2^T + L_2^T - M_1 + M_2^T + N_1$, $\Omega_{13} = G_1 A_d + A^T G_3^T - L_1 + L_3^T + M_3^T - N_1$,
$\Omega_{14} = P - G_1 + A^T G_4^T + L_4^T + M_4^T$, $\Omega_{15} = G_1 E$,
$\Omega_{22} = -(1-d_1)(Q_2 - Q_3) - M_2 - M_2^T + N_2 + N_2^T$,
$\Omega_{23} = G_2 A_d - L_2 - M_3^T - N_2 + N_3^T$, $\Omega_{24} = -G_2 - M_4^T + N_4^T$, $\Omega_{25} = G_2 E$,
$\Omega_{33} = -(1-d_1-d_2)(Q_1 + Q_3) + G_3 A_d + A_d^T G_3^T - L_3 - L_3^T - N_3 - N_3^T$,
$\Omega_{34} = -G_3 + A_d^T G_4^T - L_4^T - N_4^T$, $\Omega_{35} = G_3 E$, $\Omega_{44} = \bar{\tau} R_1 + \bar{\tau}_1 R_2 + \bar{\tau}_2 R_3 - G_4 - G_4^T$,
$\Omega_{45} = G_4 E$, $\Omega_{55} = 0$.

Simplifying Eq. (19) we have

$$\dot{V}(t) \le \bar{\xi}^T(t)\left[\bar{\Psi}_1 + \Psi_2 + \Psi_2^T + \Psi_3 + \Psi_4\right]\bar{\xi}(t) \tag{20}$$

where

$$\bar{\Psi}_1 = \begin{bmatrix} Q_1+Q_2 & 0 & 0 & P & 0 \\ * & -(1-d_1)(Q_2-Q_3) & 0 & 0 & 0 \\ * & * & -(1-d_1-d_2)(Q_1+Q_3) & 0 & 0 \\ * & * & * & \bar{\tau}R_1 + \bar{\tau}_1 R_2 + \bar{\tau}_2 R_3 & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$\Psi_4 = \Psi_5 \Xi_5 \Psi_5^T$, $\Xi_5 = \mathrm{diag}\{\bar{\tau} R_1^{-1}, \bar{\tau}_1 R_2^{-1}, \bar{\tau}_2 R_3^{-1}\}$, and Ψ_2, Ψ_3, Ψ_5 are given in Eq. (3).

Thus, we have

$$\begin{aligned} &\dot{V}(t) + y^T(t)y(t) - \gamma^2 w^T(t)w(t) \le \\ &\bar{\xi}^T(t)\left[\Psi_1 + \Psi_2 + \Psi_2^T + \Psi_3 + \Psi_6^T\Psi_6 + \Psi_4\right]\bar{\xi}(t) \end{aligned} \tag{21}$$

where Ψ_1, Ψ_2, Ψ_3, Ψ_6 are given in (3) and Ψ_4 is given in (20).

First, we consider the asymptotic stability of system Eq. (1) satisfying Eq. (2) with $w(t) = 0$. For this case, from Eq. (21) we have

$$\dot{V}(t) \le \xi^T(t)\left[\bar{\Psi}_1 + \bar{\Psi}_2 + \bar{\Psi}_2^T + \bar{\Psi}_3 + \bar{\Psi}_4\right]\xi(t) \tag{22}$$

where

$\xi(t) = \left[x^T(t) \quad x^T(t-\tau_1(t)) \quad x^T(t-\tau(t)) \quad \dot{x}^T(t)\right]^T$

$$\overline{\Psi}_1=\begin{bmatrix} Q_1+Q_2 & 0 & 0 & P \\ * & -(1-d_1)(Q_2-Q_3) & 0 & 0 \\ * & * & -(1-d_1-d_2)(Q_1+Q_3) & 0 \\ * & * & * & \overline{\tau}R_1+\overline{\tau}_1R_2+\overline{\tau}_2R_3 \end{bmatrix}$$

$$\overline{\Psi}_2=\overline{\Psi}_{21}\overline{\Psi}_{22},\ \overline{\Psi}_{21}=\begin{bmatrix} A & 0 & A_d & -I\end{bmatrix}^T,\ \overline{\Psi}_{22}=\begin{bmatrix} G_1^T & G_2^T & G_3^T & G_4^T\end{bmatrix}$$

$$\overline{\Psi}_3=\begin{bmatrix} L_1+L_1^T+M_1+M_1^T & L_2^T-M_1+M_2^T+N_1 & -L_1+L_3^T+M_3^T-N_1 & L_4^T+M_4^T \\ * & -M_2-M_2^T+N_2+N_2^T & -L_2-M_3^T-N_2+N_3^T & -M_4^T+N_4^T \\ * & * & -L_3-L_3^T-N_3-N_3^T & -L_4^T-N_4^T \\ * & * & * & 0 \end{bmatrix}$$

$$\overline{\Psi}_4=\overline{\Psi}_5\overline{\Xi}_5\overline{\Psi}_5^T,\ \overline{\Xi}_5=\mathrm{diag}\{\overline{\tau}R_1^{-1},\overline{\tau}_1R_2^{-1},\overline{\tau}_2R_3^{-1}\},\ \overline{\Psi}_5=\begin{bmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \\ L_4 & M_4 & N_4 \end{bmatrix}$$

Notice that by Schur Complement (Lemma 2), Eq. (3) implies $\overline{\Psi}_1+\overline{\Psi}_2+\overline{\Psi}_2^T+\overline{\Psi}_3+\overline{\Psi}_4<0$ and hence $\dot{V}(t)<0$. Therefore, we can conclude that system (1) satisfying (2) with $w(t)=0$ is asymptotically stable.

Now, we consider stable system with H_∞ performance, that $\|y\|_2<\gamma\|w\|_2$ for all nonzero $w\in L_2[0,\infty)$ under zero initial condition.

By Schur Complement (Lemma 2), Eq. (3) guarantees $\Psi_1+\Psi_2+\Psi_2^T+\Psi_3+\Psi_6^T\Psi_6+\Psi_4<0$, where Ψ_1, Ψ_2, Ψ_3, Ψ_6 are given in Eq. (3) and Ψ_4 is given in Eq. (20). Therefore, from Eq. (21) we have,

$$y^T(t)y(t)-\gamma^2w^T(t)w(t)+\dot{V}(t)<0 \tag{23}$$

for all nonzero $w\in L_2[0,\infty)$. Under zero initial condition, we have $V(0)=0$ and $V(\infty)\geq 0$. Taking integration on both sides of Eq. (23) yields,

$$\int_0^\infty y^T(t)y(t)dt-\int_0^\infty \gamma^2w^T(t)w(t)dt+\int_0^\infty \dot{V}(t)dt<0$$

$$\|y\|_2^2-\gamma^2\|w\|_2^2<0$$

Thus,

$\|y\|_2 < \gamma\|w\|_2$ for all nonzero $w \in L_2[0,\infty)$ and the proof is complete. □

Theorem 1 provides condition in terms of LMI (3) that can be numerically solved using any standard LMI solver, e.g. [11].

Remark 1. A different LMI condition for H_∞ performance analysis derived in Theorem 2 of [7]. Notice that, the LMI condition in Theorem 1 in this paper involves the same number of unknown parameters than those in [7].

Remark 2. Due to the condition of network transmission, the delays $\tau_1(t)$ and $\tau_2(t)$ in Eq. (1) may not be identical [5] and another LMI condition may be obtained by assuming different properties of those delays.

Using the result derived in this paper, we further consider the model of Networked Control Systems described in [7]. In this case, the two additive delays $\tau_1(t)$ and $\tau_2(t)$ have very different properties in that $\tau_1(t)$ and $\tau_2(t)$ are assumed to be constant and non-differentiable, respectively, i.e.

$$\tau_1(t) \equiv \bar{\tau}_1 < \infty, 0 \le \tau_2(t) \le \bar{\tau}_2 < \infty \qquad (24)$$

We then have the following corollary.

Corollary 1. System Eq. (1) satisfying Eq. (24) either
(i) asymptotically stable with $w = 0$, or
(ii) stable with H_∞ disturbance attenuation level γ ($w \neq 0$)
if there exist matrices $P = P^T > 0$, $Q_2 = Q_2^T > 0$, $R_1 = R_1^T > 0$, $R_2 = R_2^T > 0$, $R_3 = R_3^T > 0$ and G_i, L_i, M_i, N_i, $i = 1,...,4$ are free matrices satisfying,

$$\begin{bmatrix} \bar{\bar{\Psi}}_1 + \Psi_2 + \Psi_2^T + \Psi_3 + \Psi_6^T \Psi_6 & \Psi_5 \\ * & \Xi_5 \end{bmatrix} < 0 \qquad (25)$$

where,

$$\bar{\bar{\Psi}}_1 = \begin{bmatrix} Q_2 & 0 & 0 & P & 0 \\ * & -Q_2 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & \bar{\tau}R_1 + \bar{\tau}_1 R_2 + \bar{\tau}_2 R_3 & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix}$$

Other parameters are given in Eq. (3).

Proof.

Define a Lyapunov-Krasovski functional,

$$V(t) = V_1(t) + V_2(t) + V_3(t)$$

where $V_1(t)$ and $V_3(t)$ are given in Eqs. (5) and (7), respectively and

$$V_2(t) = \int_{t-\bar{\tau}_1}^{t} x^T(s) Q_2 x(s) ds$$

Then, the proof is derived along similar lines as in the proof of Theorem 1. □

4 Illustrative Example

Suppose the system matrices A, A_d, C, C_d, E, F in Eq. (1) are given as follows [7],

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},\ A_d = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},\ E = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ C_d = \begin{bmatrix} 1 & 0 \end{bmatrix},\ F = 0.5.$$

The parameters for delay are given by $d_1 = 0.1$ and $d_1 = 0.8$.

We assume the delay upper bound $\bar{\tau}_1$ and $\bar{\tau}_2$. Our objective is, to find the minimum guaranteed H_∞ performance, γ_{min}, that makes condition in Theorem 1 feasible. Firstly, we assume $\bar{\tau}_1 = 1$ and $\bar{\tau}_2 = 0.1$. By solving LMI (3), we get the minimum guaranteed H_∞ performance, γ_{min} = 1.919. Comparison for different cases with $\bar{\tau}_1$ and $\bar{\tau}_2$ varies is provided in Table 1. It shows that the smaller the upper bound of the delay system, the smaller γ_{min} we get.

Table 1 Minimum guaranteed H_∞ performance, γ_{min}, for different cases of $\bar{\tau}_1$ and $\bar{\tau}_2$.

$\bar{\tau}_1$ **(s)**	**1**			**1.2**			**1.5**		
$\bar{\tau}_2$ (s)	0.1	0.2	0.3	0.1	0.2	0.3	0.1	0.2	0.3
γ_{min}	1.919	3.308	7.841	2.632	5.276	14.109	4.419	10.222	36.609

5 Conclusion

In this paper, we have investigated the asymptotic stability of continuous time system with two additive time-varying delays, with H_∞ disturbance attenuation level γ. By exploiting Lyapunov-Krasovski functional and introducing free weighting matrix variables, the stability condition and H_∞ performance were derived by using linear matrix inequality (LMI) techniques. An illustrative example is provided to validate the analysis method.

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