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Analysis of The Rosenzweig-MacArthur Predator-Prey Model with Anti-Predator Behavior

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ABSTRACT

This paper discusses the analysis of the Rosenzweig-MacArthur *predator-prey* model with *anti-predator* behavior. The analysis is started by determining the equilibrium points, existence, and conditions of the stability. Identifying the type of Hopf bifurcation by using the divergence criterion. It has shown that the model has three equilibrium points, i.e., the extinction of population equilibrium point (E_0), the non-*predatory* equilibrium point (E_1), and the co-existence equilibrium point (E_2). The existence and stability of each equilibrium point can be shown by satisfying several conditions of parameters. The divergence criterion indicates the existence of the supercritical Hopf-bifurcation around the equilibrium point E_2. Finally, our model's dynamics population is confirmed by our numerical simulations by using the 4th-order Runge-Kutta methods.

Keywords: Rosenzweig-MacArthur; predator-prey model; anti-predator behaviour; Hopf Bifurcation; divergence criterion; equilibrium point.

INTRODUCTION

Population dynamics are the most interesting research in mathematical biology which discusses the interactions that occur between *prey* and *predator* in a particular ecosystem [1]. This interaction has implemented to a simple mathematical model known as the Lotka-Volterra *predator-prey* model [2].

In a mathematical model, the predation process (interaction between *prey* and *predator*) is expressed in some form that is known as a functional response. This functional response has classified three functions, i.e. Holling-Type I, Holling-Type II, and Holling-Type III where each type determine the characteristic of the *predator* [3]. On the progress, Rosenzweig and MacArthur modifying the Lotka-Volterra *predator-prey* model with the assumption the attack rate of *predator* increases at a decreasing rate with *prey* density until it becomes constant due to satiation which is affected by Holling-Type II functional response [4]. Further, some modified of Lotka-Volterra *predator-prey* model by considering the infectious disease [5]-[7].

Several research has discussed the modification of the Rosenzweig-MacArthur *predator-prey* model [8][9] is introduced *predator* foraging facilitation into Holling-Type II functional response. Furthermore, the Rosenzweig-MacArthur model has modified with various factors, e.g. the stage-structure [10][11], the refuge effect [12][13], the harvesting to one or more population [14][15]. From several studies described above, no one

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considering *anti-predator* behavior factors.

In this article, the Rosenzweig-MacArthur *predator-prey* model by [6] modified considering *anti-predator* behavior factors [16]. These factors can be considered in the model because the dynamics of the model will be very complex when the *prey* population prefers to defending and provide resistance when the predation process is occurring. The structure of this paper is as follows. In the next section, the methods in our work are described. Then, the analysis of the model has been discussed. Finally, a brief conclusion of our work is given.

METHODS

The dynamics of the model is analyzed by carrying out the following steps:

1. Modifying the Rosenzweig-MacArthur *predator-prey* model considering *anti-predator* behavior factors.
2. Simplifying the model by using non-dimensional to reduce the number of parameters and solving the equilibrium points of the model.
3. Identifying the existence, local stability, and global stability of the equilibrium points.
4. Identifying the Hopf-bifurcation type by using the divergence criterion.
5. Demonstrated the numerical simulations of the model to describe the analysis results by using the 4th-order Runge-Kutta method.

RESULTS AND DISCUSSION

Mathematical Model

In this article, the mathematical model is formulated based on the following assumptions:

1. The *prey* population is assumed to grow logistically with an intrinsic growth rate of r and carrying capacity of the environment of K and reduced due to the predation process.
2. The *predator* population is assumed to grow due to the predation process. c is the conversion rate of the consumed *prey* into *predator* births.
3. The predation process follows Holling-Type II functional response which is affected by the *encounter rate* function where there is foraging facilitation of *predator* $(w = 0)$, a is the saturated rate of the *predator*, b is coefficient interaction on both population and h is the *predator* time handling.
4. m is the mortality of *predators*.
5. η is the *anti-predator* behavior.

From the following assumptions above, the dynamics of the model can be represented by the following set of differential equations:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\frac{(a-b)xy}{y+h(a-b)x}\\ \frac{dy}{dt} &= \frac{c(a-b)xy}{y+h(a-b)x}-my-\eta xy\end{aligned} \tag{1}$$

Where x and y are respectively the densities of *prey* and *predator* population at time t and $x(0), y(0) > 0$.

To simplify our analysis, we reduce the number of parameters in system (1) by using the following parameter scales [17]:

$$x \to xK, \qquad y \to y(a-b)Kh, \qquad t \to \frac{t}{r}$$

We obtain the following non-dimensional model

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - \frac{\alpha xy}{x+y} \\ \frac{dy}{dt} &= \frac{\beta xy}{x+y} - \gamma y - \delta xy \end{aligned} \tag{2}$$

where

$$\alpha = \frac{(a-b)}{r}, \qquad \beta = \frac{c}{hr}, \qquad \gamma = \frac{m}{r}, \qquad \delta = \frac{\eta K}{r}$$

Existence and Stability Analysis of Equilibrium Points

In this section, the equilibrium point of model (2) is obtained by solving [18]:

$$\begin{aligned} x(1-x) - \frac{\alpha xy}{x+y} &= 0 \\ \frac{\beta xy}{x+y} - \gamma y - \delta xy &= 0 \end{aligned} \tag{3}$$

Thus, from the system (3), we obtain the following equilibrium points, i.e.:

1. A trivial equilibrium point $E_0 = (0,0)$, always exists.
2. A non-*predator* equilibrium point $E_1 = (1,0)$, always exists too.
3. A co-existence equilibrium point $E_2 = (x^*, y^*)$, where

$$x^* = \frac{\beta - \alpha\beta + \alpha\gamma}{\beta - \alpha\delta}, \qquad y^* = \frac{(\beta - \alpha\beta + \alpha\gamma)(\beta - \gamma - \delta)}{(\beta - \alpha\delta)(\gamma + \delta - \alpha\delta)}$$

which exists if

$$\beta > \alpha(\beta - \gamma), \qquad \gamma + \delta < \alpha\delta < \beta$$

Now, study the local stability of the dynamics of the system (3) around each of equilibrium point. The Jacobian matrix from the system (3) is determined as [19]:

$$J_{(x,y)} = \begin{pmatrix} 1 - 2x - \frac{\alpha y}{x+y} + \frac{\alpha xy}{(x+y)^2} & -\frac{\alpha x}{x+y} + \frac{\alpha xy}{(x+y)^2} \\ \frac{\beta y}{x+y} - \frac{\beta xy}{(x+y)^2} - \delta y & \frac{\beta x}{x+y} - \frac{\beta xy}{(x+y)^2} - \gamma - \delta x \end{pmatrix} \tag{4}$$

By evaluating this Jacobian matrix (4) at each equilibrium point, we obtain the local stability properties of E_0, E_1, and E_2 as follows.

Theorem 1. *The trivial equilibrium point E_0 always unstable (saddle).*
Proof:
The Jacobian matrix (4) evaluated in equilibrium point E_0 is given by

$$J_{(E_0)} = \begin{pmatrix} 1 & 0 \\ 0 & -\gamma \end{pmatrix}$$

So, by solving the characteristic equation, we obtained the eigenvalues of $J_{(E_0)}$ is $\lambda_1 = 1$ and $\lambda_2 = -\gamma$. It means $\lambda_1 > 0$ and $\lambda_2 < 0$. Therefore, stability of equilibrium point E_0 is unstable (*saddle*).■

Theorem 2. *If $\delta > \beta - \gamma$, then the non-predatory equilibrium point E_1 of system (2) is locally asymptotically stable.*
Proof:
The Jacobian matrix (4) evaluated in equilibrium point E_1 is given by

$$J_{(E_1)} = \begin{pmatrix} -1 & -\alpha \\ 0 & \beta - \gamma - \delta \end{pmatrix}$$

So, by solving the characteristic equation, we obtained the eigenvalues of $J_{(E_1)}$ is $\lambda_1 = -1$ and $\lambda_2 = \beta - \gamma - \delta$. It means $\lambda_1 < 0$. Therefore, if $\delta > \beta - \gamma$ then each the eigenvalues of $J_{(E_1)}$ are negatif, and E_1 is locally asymptotically stable.■

Theorem 3. *The co-existence equilibrium point E_2 is locally asymptotically stable if the conditions below are satisfied*

$$\delta^2 < \frac{\Theta + \Upsilon}{\mathrm{Z}}$$

Proof:
The Jacobian matrix (4) evaluated in equilibrium point E_1 is given by

$$J_{(E_2)} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

Where

$$M_{11} = \frac{-\beta^2 + \alpha\beta^2 - \alpha\gamma^2 - 2\alpha\delta(\alpha - 1)(\beta - \gamma) - \alpha\delta^2 + \alpha^2\delta^2}{(\beta - \alpha\delta)^2}$$

$$M_{12} = -\frac{\alpha(\gamma + \delta - \alpha\delta)^2}{(\beta - \alpha\delta)^2}$$

$$M_{21} = \frac{(\beta - \gamma - \delta)\left(\beta^2\gamma + \alpha^2\gamma\delta^2 - \beta(\gamma^2 + 2\gamma\delta + \delta^2(\alpha - 1)^2)\right)}{(\beta - \alpha\delta)^2}$$

$$M_{22} = -\frac{\beta(\beta - \gamma - \delta)(\gamma + \delta - \alpha\delta)}{(\beta - \alpha\delta)^2}$$

By solving the characteristic equation, we obtained the eigenvalues of $J_{(E_2)}$ is

$$\lambda_{1,2} = \frac{1}{2} \cdot \frac{1}{(\beta - \alpha\delta)^2}(A \pm B)$$

Where

$$A = \mathrm{Z}\delta^2 - \Theta - \Upsilon \qquad \text{and} \qquad B = \Psi^2 - \alpha\Omega$$

With

$$\mathrm{Z} = (\alpha^2 - \alpha + \beta - \alpha\beta)$$
$$\Theta = \delta\big(\beta(\beta - 2\gamma) + 2\alpha^2(\beta - \gamma) - \alpha(\beta^2 + 2(\beta - \gamma) - \beta\gamma)\big)$$
$$\Upsilon = \beta^2(\gamma - \alpha + 1) + \gamma^2(\alpha + \beta)$$
$$\Psi = \big(\beta^2 - \alpha\beta^2 + \alpha\gamma^2 + 2\alpha\delta(\alpha - 1)(\beta - \gamma) - \alpha\delta^2(\alpha - 1) - \beta(\beta - \gamma - \delta)(\gamma + \delta - \alpha\delta)\big)$$
$$\Omega = 4(\beta - \gamma - \delta)(\gamma + \delta - \alpha\delta)\big(\beta^2\gamma + \alpha^2\gamma\delta^2 - \beta((\alpha - 1)^2\delta^2 + \gamma^2 + 2\gamma\delta)\big)$$

According to (), the stability of equilibrium point E_2 depending on the value of A. If $A < 0$, we obtained:

$$\mathrm{Z}\delta^2 - \Theta\delta - \Upsilon < 0$$
$$\mathrm{Z}\delta^2 < \Theta\delta + \Upsilon$$
$$\delta^2 < \frac{\Theta + \Upsilon}{\mathrm{Z}}$$

By the conditions above, the stability of equilibrium point E_2 is locally asymptotically stable.■

Next, study the global stability of the dynamics of the system (3) around equilibrium point E_2. We obtain the global stability properties of E_2 by using the Lyapunov function [20] as follows.

Theorem 4. *The co-existence equilibrium E_2 is globally asymptotically stable if the conditions below are satisfied:*

$$x^* < \frac{(\alpha - \beta + \gamma + \delta)(\gamma + \delta - \alpha\delta)}{\alpha(\gamma + \delta - \alpha\delta) - (\beta - \gamma - \delta)^2}$$

***Proof*:**
Define a Lyapunov function as follows

$$V(x, y) = \left[x - x^* - x^* \ln\left(\frac{x}{x^*}\right)\right] + \left[y - y^* - y^* \ln\left(\frac{y}{y^*}\right)\right]$$

By using the function $\dot{V} < 0, \forall\, (x, y) \in \mathbb{R}_2^+$, we obtain:

$$\frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial V}{\partial y} \cdot \frac{\partial y}{\partial t} \leq 0$$

$$\left(1 - \frac{x^*}{x}\right)\left(x(1 - x) - \frac{\alpha x y}{x + y}\right) + \left(1 - \frac{y^*}{y}\right)\left(\frac{\beta x y}{x + y} - \gamma y - \delta x y\right) \leq 0$$

$$\left(\frac{(1 - x)(x + y) - \alpha y}{x + y}\right)(x - x^*) + \left(\frac{\beta x - \gamma(x + y) - \delta x(x + y)}{x + y}\right)(y - y^*) \leq 0$$

For $(x, y) \in \mathbb{R}_2^+$, we obtain:

$$-\alpha + \alpha x^* + \beta - \gamma - \delta - (\beta - \gamma - \delta)y^* < 0$$

$$-\alpha + \alpha x^* + \beta - \gamma - \delta - x^* \frac{(\beta - \gamma - \delta)^2}{(\gamma + \delta - \alpha\delta)} < 0$$

$$x^* \left(\frac{\alpha(\gamma + \delta - \alpha\delta) - ((\beta - \gamma - \delta)^2)}{(\gamma + \delta - \alpha\delta)} \right) < \alpha - \beta + \gamma + \delta$$

$$x^* < \frac{(\gamma + \delta - \alpha\delta)(\alpha - \beta + \gamma + \delta)}{\alpha(\gamma + \delta - \alpha\delta) - ((\beta - \gamma - \delta)^2)}$$

By the conditions above, the stability of equilibrium point E_2 is globally asymptotically stable.■

Analysis of Hopf Bifurcation Type

In this section, we'll define the Hopf-bifurcation type by using the divergence criterion [21]. System (3) underwent a Hopf-bifurcation when it satisfies the following conditions:

$$\delta^2 < \frac{\Theta + \Upsilon}{Z} \qquad \text{and} \qquad \alpha > \frac{\Psi^2}{\Omega}$$

To determine the Hopf-bifurcation type of system (3) on equilibrium point E_2, then we formed a new system. Let $\phi(x, y)$ is a divergence of (af, ag). We obtain the coefficient value of $a(x, y)$ of the system (3) when the parameter value $\alpha = 2, \beta = 0.79, \gamma = 0.5$, and $\delta = 0.0186$ with equilibrium point $E_2^* = (0.279; 0.157)$ as follows:

$$a(x, y) = 1 + 6.956x + 13{,}386y - 6.77x^2 + 32.968xy + 55.507y^2$$

So that a new system is obtained:

$$\begin{aligned} z(x, y) &= (1 + 6.956x + 13{,}386y - 6.77x^2 + 32.968xy + 55.507y^2) \\ &\quad \left(x(1 - x) - \frac{\alpha xy}{x + y} \right) \\ w(x, y) &= (1 + 6.956x + 13{,}386y - 6.77x^2 + 32.968xy + 55.507y^2) \\ &\quad \left(\frac{\beta xy}{x + y} - \gamma y - \delta xy \right) \end{aligned} \tag{5}$$

By linearizing system (4), we obtained:

$$J_{(E_2^*)} = \begin{pmatrix} 1.337 & -6.002 \\ 0.732 & -1.337 \end{pmatrix}$$

By solving the characteristic equation, we obtained the eigenvalues of $J_{(E_2^*)}$ is

$$\lambda_{1,2} = \pm 1.615i$$

For a system (5) to obtain the eigenvalues of conjugate complex numbers, then we can analyze the Hopf-bifurcation of system (3) type by looking at the divergence value of system (3). We obtained:

$$\phi_{xx}(E_2^*) = -21.109$$

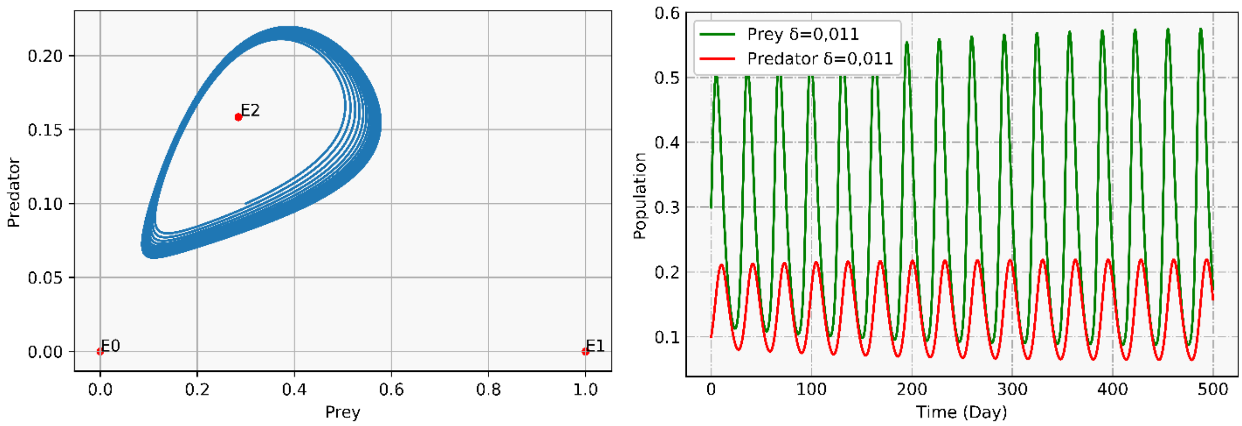
Based on the divergence value above, a stable *limit cycle* appears in the system (3). Therefore, system (3) underwent a Supercritical Hopf-bifurcation.

Numerical Simulations

In this section, the numerical simulation is solved using the 4th-order Runge-Kutta method [22] with initial conditions and some values of the parameters. We choose the following set of parameter values:

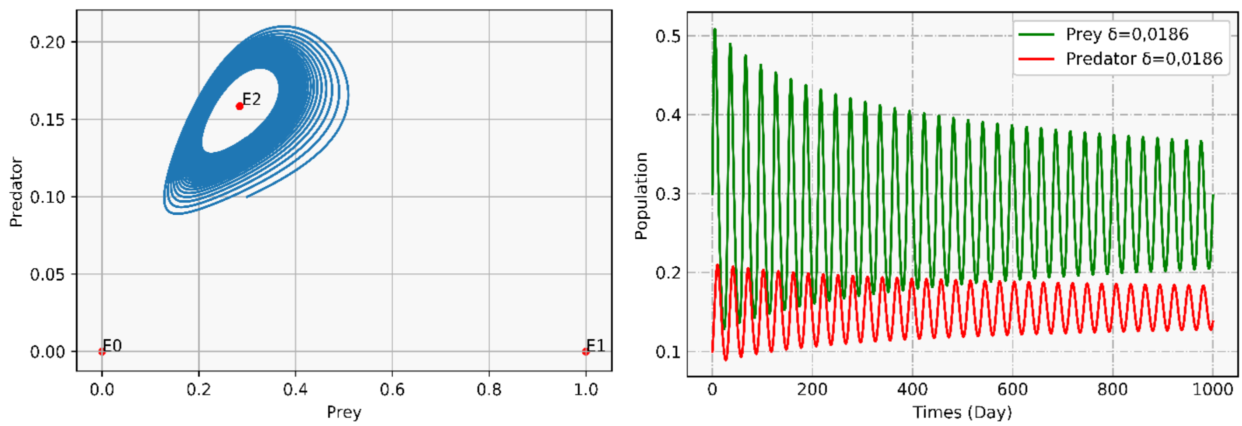
$$\alpha = 2, \qquad \beta = 0.79, \qquad \gamma = 0.5$$

With different parameter control values as follows $\delta_1 = 0.011$, $\delta_2 = 0.0186$ and $\delta_3 = 0.026$. We using the initial condition is $x(0) = 0.3$ and $y(0) = 0.3$.



(a) **(b)**

Figure 1. (a) Phase Portrait of Case 1 and **(b)** Time-Series Portrait

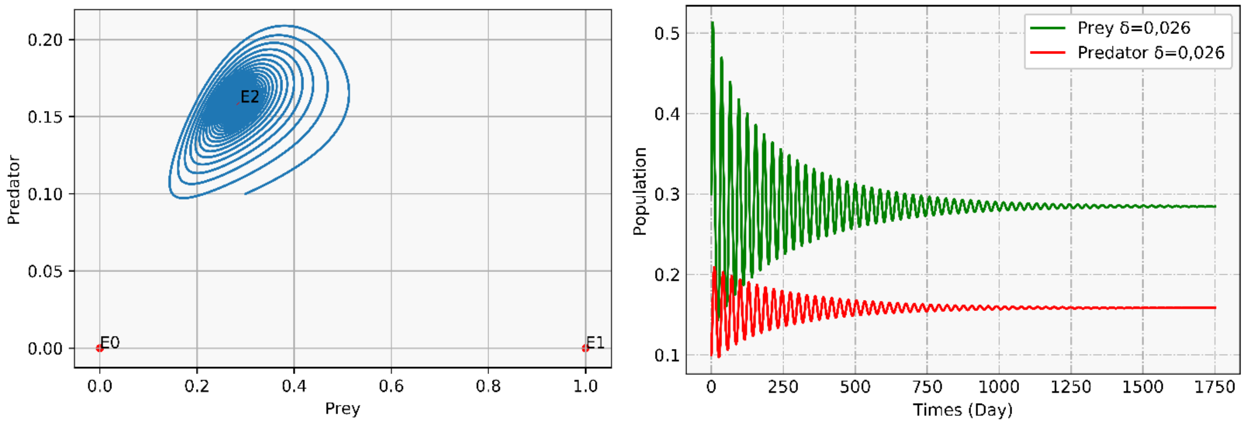


(a) **(b)**

Figure 2. (a) Phase Portrait of Case 2 and **(b)** Time-Series Portrait

In case 1, we obtained the dynamics of the solution on the system (3) with parameter control values $\delta_1 = 0.011$. Based on **figure**

1(a), the trivial equilibrium point $E_0 = (0,0)$ is unstable (*saddle*) with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -0.5$. This coincides with **Theorem 1**. The non-*predator* equilibrium point $E_1 = (1,0)$ is unstable (*saddle*) with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 0.279$. This coincides with **Theorem 2** on condition $\delta < \beta - \gamma$. The co-existence equilibrium point $E_2 = (0.273; 0.156)$ is unstable (*spiral*) with eigenvalues $\lambda_{1,2} = 0.003 \pm 0.220i$. This coincides with **Theorem 3** on condition $\delta^2 < \frac{\Theta+\Upsilon}{Z}$. Based on **figure 1(b)**, the *prey* population and *predator* population have increased and decreased of total populations. The case continuously oscillates with a greater deviation value. Hence, both population is unstable to a specific point.



(a) **(b)**

Figure 3. (a) Phase Portrait of Case 3 and **(b)** Time-Series Portrait

In case 2, we obtained the dynamics of the solution on the system (3) with parameter control values $\delta_1 = 0.0186$. Based on **figure 2(a)**, the trivial equilibrium point $E_0 = (0,0)$ is unstable (*saddle*) with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -0.5$. This coincides with **Theorem 1**. The non-*predator* equilibrium point $E_1 = (1,0)$ is unstable (*saddle*) with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 0.271$. This coincides with **Theorem 2** on condition $\delta < \beta - \gamma$. The co-existence equilibrium point $E_2 = (0.279; 0.157)$ is center (*spiral*) with eigenvalues $\lambda_{1,2} = \pm 0.220i$. This coincides with **Theorem 3** on condition $\delta^2 = \frac{\Theta+\Upsilon}{Z}$. Based on **figure 2(b)**, the oscillations that occur have a smaller deviation value. This condition explains that there is a stability transition from unstable to stable to a specific point. This stability transition has led to the appearance of Hopf-bifurcation.

In case 3, we obtained the dynamics of the solution on the system (3) with parameter control values $\delta_1 = 0.026$. Based on **figure 3(a)**, the trivial equilibrium point $E_0 = (0,0)$ is unstable (*saddle*) with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -0.5$. This coincides with **Theorem 1**. The non-*predator* equilibrium point $E_1 = (1,0)$ is unstable (*saddle*) with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 0.264$. This coincides with **Theorem 2** on condition $\delta < \beta - \gamma$. The co-existence equilibrium point $E_2 = (0.285; 0.159)$ is stable (*spiral*) with eigenvalues $\lambda_{1,2} = -0.003 \pm 0.220i$. This coincides with **Theorem 3** on condition $\delta^2 > \frac{\Theta+\Upsilon}{Z}$. Based on **figure 3(b)**, the dynamics between prey and predator begin to stabilize at 1500 days to a specific point.

CONCLUSIONS

The Rosenzweig-MacArthur *predator-prey* model with *anti-predator* behavior has been studied. From the analysis of system (2), we obtain three equilibrium points, i.e., the trivial equilibrium point (E_0), the non-*predatory* equilibrium point (E_1), and the co-existence equilibrium point (E_2). The local stability conditions of each equilibrium point have been appointed, and the global stability conditions of the co-existence equilibrium point (E_2) have been obtained. Our analysis also showed that the model occurs a Supercritical Hopf-bifurcation by using the divergence criterion. Numerical analytic has been simulated to verify the theoretical results. No one extinction matters in any population.

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